

On conformal reflections in compactified phase space

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ABSTRACT: Some results from arguments of research dealt with R. Raczka are exposed and extended. In particular new arguments are brought in favor of the conjecture, formulated with him, that both space-time and momentum may be conformally compactified, building up a compact phase space of automorphism for the conformal group, where conformal reflections determine a convolution between space-time and momentum space which may have consequences of interest for both classical and quantum physics.

1. Introduction

In 1931 P.M.A. Dirac wrote [1]: *“There are at present fundamental problems in theoretical physics ... whose solution will presumably require a more drastic revision of our fundamental concepts than any that have gone before. Quite likely these changes will be so great that it will be beyond the power of human intelligence to get the necessary new ideas by direct attempts to formulate the experimental data in mathematical terms. The theoretical work in the future will therefore have to proceed in a more indirect way. The most powerful method of advance that can be suggested at present is to employ all the resources of pure mathematics in attempts to perfect and generalize the mathematical formalism that forms the existing basis of theoretical physics, and after each success in this direction, to try to interpret the new mathematical features in terms of physical entities ...”*. Later he added: *“It seems that, if one is working from the point of view of getting beauty in one’s equations, and if one has really sound insight, one is on a sure line of progress”*.

Dirac started himself the process of research in abstract mathematics, aimed at the understanding of physical quantum phenomena, which brought him to discover his beautiful spinor field equation which, not only explained the origin of electron spin, but even anticipated the discovery of new, unexpected physical phenomena; as those deriving from the existence of antimatter. He was certainly a forerunner and now, after 67 years, when looking at the recent developments in theoretical physics, where some branches like quantum groups, noncommutative geometry, deal mainly with subjects of pure mathematics, his words sound like a prophecy.

Having in mind his recommendations, an undeniable source of beauty, of relevance for quantum physical phenomena, may certainly be found in the E. Cartan work on simple spinor geometry [2]. E. Cartan stressed specially two concepts.

1. Vectors of euclidean spaces may be conceived as bilinearly composed by spinors.
2. Rotations in vector spaces may be decomposed in products of reflections.

Notoriously, the main geometrical tool for the description of the phenomena of classical mechanics is euclidean geometry in pseudo euclidean vector spaces and rotation therein. Following E. Cartan, spinors and reflection may be conceived as the elementary constituents of these.

The study of covariance of the equations of motion with respect to rotation groups has been one of the main mathematical instrument of research in classical mechanics in the last and also in the present century, when covariance of Maxwell’s equations with respect to the Lorentz-Poincaré groups has brought Einstein to the discovery of special relativity and of the geometrical structure of space-time: $\mathbb{M} = \mathbb{R}^{3,1}$.

The importance of reflection groups, instead, has been only recognized after the advent of quantum mechanics. In fact space-time reflection group play an important

role in the explanation of quantum phenomena. Precisely space reflections allow us to define the concept of parity (not conserved in weak decays) while time inversion allows us to understand the existence of antimatter. The most appropriate space for dealing with such reflections seems to be spinor space.

It is known since 1909 that Maxwell's equations are also covariant with respect to the conformal group. A discovery which has also brought to the conjecture that Minkowski space-time may be conformally compactified, and represented by a particular realization of an homogeneous space of the conformal group. With R. Raczka, somehow in line with Dirac's recommendations, we have observed that another realization of that homogeneous space could well represent conformally compactified momentum space which, together with compactified space-time, could then build up a compact phase space where both the concept of infinity and that of infinitesimal, of difficult, if not impossible, self consistent mathematical definition, would not be needed, allowing then not only a rigorous mathematical formulation of theoretical physics, but also the elimination of the main difficulties encountered by quantum physics: the ones of so called infrared and ultraviolet divergences.

The conformal group, in which the Lorentz-Poincaré group is contained as a subgroup, has a reflection group which contains, beside space-time reflections, mentioned above, also two additional reflections, called conformal reflections. However, somehow surprisingly, despite the relevance of conformal covariance for the understanding of several physical phenomena concerning massless systems, these reflections have failed, up to now, to manifest their role in physics.

One of purposes of this paper is to review some of the results obtained in the work with R. Raczka and to outline some follow up of our thinking, in particular on the possible role in physics of conformal reflections, obtained after he left us.

2. Compact phase space.

In a similar way as, from Maxwell's equations Lorentz covariance, Minkowski derived the pseudo euclidean structure of space-time: $\mathbb{M} = \mathbb{R}^{3,1}$, from their conformal covariance Veblen derived and adopted [3] the conformally compactified structure \mathbb{M}_c of \mathbb{M} :

$$\mathbb{M}_c = \frac{S^3 \times S^1}{\mathbb{Z}_2} , \quad (2.1)$$

in which Minkowski space-time \mathbb{M} is densely contained (which means: to every point of \mathbb{M}_c there correspond one point of \mathbb{M} ; to a submanifold of \mathbb{M}_c , of dimension 3, there corresponds the points of \mathbb{M} at infinity) afterwards also adopted by several authors [4].

We will show now that the same argument which induces to postulate that \mathbb{M} is compactified induces also to postulate that its Fourier dual momentum space $\mathbb{P} = \mathbb{R}^{3,1}$ is conformally compactified as well.

In fact the conformal group may be linearly represented by $O(4, 2)$ acting in $\mathbb{R}^{4,2}$. Let us consider the equations for the corresponding Weyl spinors or twistors:

$$\sum_{a=1}^6 p_a \gamma^a (1 \pm \gamma_7) \pi = 0 \quad , \quad (2.2)$$

where $p_a \in \mathbb{R}^{4,2}$, γ^a are the generators of the Clifford algebra $\mathbb{C}\ell(4, 2)$, γ_7 its volume elements and π is a vector of the spinor space S defined by $\text{End}S = \mathbb{C}\ell(4, 2)$.

Eq. (2.2) for non zero twistors $\pi_{\pm} = \frac{1}{2} (1 \pm \gamma_7) \pi$, implies $p_a p^a = 0$, and therefore the directions of p_a form the projective quadric \mathbb{P}_c :

$$\mathbb{P}_c = \frac{S^3 \times S^1}{\mathbb{Z}_2}. \quad (2.3)$$

It may be easily seen that eq. (2.2) contains the equations:

$$\sum_{\mu=0}^3 p_{\mu} \gamma^{\mu} \varphi_{\pm} = 0 \quad , \quad (2.4)$$

where φ_{\pm} represent Weyl spinors associated with $\mathbb{C}\ell(3, 1)$, of which γ^{μ} are the generators. Eq. (2.4) is the Weyl equation in momentum space $\mathbb{P} = \mathbb{R}^{3,1}$, Fourier dual of Minkowski space $\mathbb{M} = \mathbb{R}^{3,1}$, from it is easy to obtain [5] the equations:

$$p_{\mu} F_{+}^{\mu\nu} = 0 ; \quad p_{\mu} F_{-}^{\mu\nu} = 0 \quad , \quad (2.5)$$

where

$$F_{\pm}^{\mu\nu} = \langle \varphi_{\pm}^{\dagger} [\gamma^{\mu}, \gamma^{\nu}] \varphi_{\pm} \rangle \quad .$$

Eq.s (2.5) represent the homogeneous Maxwell's equations, in momentum space $\mathbb{P} = \mathbb{R}^{3,1}$, which then results densely contained in \mathbb{P}_c of eq. (2.3).

This short-cut derivation of Maxwell's equations in momentum space from twistors equations indicates that, in so far, Maxwell's equations conformal covariance implies the conformal compactification \mathbb{M}_c , as given in eq. (2.1), of Minkowski space \mathbb{M} , their derivability from twistors equations in momentum space implies the conformal compactification \mathbb{P}_c of momentum space \mathbb{P} , as given in eq. (2.3), as well.

The resulting phase space will then be compact and consequently any field theory formulated in such a compact phase space should, a priori, expected to be free from both infrared and ultraviolet divergences.

The main problem will be to define, for every function $f(x)$ taking values in \mathbb{M}_c , a transform to a function $F(k)$, taking values in \mathbb{P}_c , such that in the flat limit (radiuses of S^3 and S^1 going to infinity) it identifies with the standard Fourier transform correlating \mathbb{M} and its dual \mathbb{P} .

The problem may be solved [6] for the two-dimensional case $\mathbb{M} = \mathbb{R}^{1,1} = \mathbb{P}$, for which:

$$\mathbb{M}_c = \frac{S^1 \times S^1}{\mathbb{Z}_2} = \mathbb{P}_c \quad ; \quad (2.6)$$

one needs only to inscribe in each S^1 a regular polygon with $2N = 2\pi RK$ vertices, where R (of dimension $[\ell]$: length) and K (of dimension $[\ell^{-1}]$) are the radiuses of the \mathbb{M}_c and \mathbb{P}_c circles respectively. They define in \mathbb{M}_c and \mathbb{P}_c two lattices: $M_L \subset \mathbb{M}_c$; $P_L \subset \mathbb{P}_c$ which are Fourier dual. Indicating in fact with $f(x_{nm})$ a function taking values in M_L and with $F(k_{\rho\tau})$ a function taking values in P_L we have :

$$f(x_{nm}) = \frac{1}{2\pi R^2} \sum_{\rho, \tau = -N}^{N-1} \varepsilon^{(n\rho - m\tau)} F(k_{\rho\tau}) \quad , \quad (2.7)$$

$$F(k_{\rho\tau}) = \frac{1}{2\pi K^2} \sum_{n, m = -N}^{N-1} \varepsilon^{-(n\rho - m\tau)} f(x_{nm}) \quad ,$$

where $\varepsilon = e^{i\frac{\pi}{N}}$ is the $2N$ -root of unity.

Eq.s (2.7), in the limit $R, K \rightarrow \infty$ may be easily identified with the standard Fourier transforms in $\mathbb{M} = \mathbb{R}^{1,1} = \mathbb{P}$.

It is obvious that in M_L and P_L any field theory will be free from both infrared and ultraviolet divergences.

In the realistic, four dimensional case, since, in principle, the concept of infinity and infinitesimal should not be realizable, one could expect again that phase space should restrict to discrete and fine lattices which however do not seem to be obtainable with standard mathematical algorithms [7], through which, instead one may anticipate some aspects of the convergences of field theories in conformally compactified space-time and momentum space.

In fact it is known that the space \mathbb{P}_c , given by eq. (2.3), is conformally flat i.e

$$g_{\mu\nu}(p) = \Omega^2(p) \eta_{\mu\nu} ,$$

where $\Omega(p)$ is the conformal factor and $\eta_{\mu\nu}$ is the metric tensor of flat $\mathbb{R}^{3,1}$. $\Omega(p)$ may be obtained ¹ by adopting the Dirac six-dimensional formalism: $p_\mu = P_\mu / (P_5 + P_6)$ and, as shown in reference [8], it provides a convergence factor since:

$$\lim_{|p|^2 \rightarrow \infty} \Omega^2(p) = \frac{M^4}{p^4} \quad , \quad (2.8)$$

where M is a mass scale ($c = 1$).

In particular for the de Sitter subgroup $SO(4, 1)$ (obtained for $P_6 = 1$):

$$\Omega(p) = \frac{4M^2}{4M^2 + p^2} \quad , \quad (2.9)$$

¹It is interesting to observe that $\Omega(p)$ may be rigorously set in the form $\Omega(p) = M^2/p_W^2$, where p_W^2 is the Wick rotated p^2 , that is $p_W^2 = p_1^2 + p_2^2 + p_3^2 + p_0^2$ which ensures the non singularity of $\Omega(p)$, which instead is not guaranteed by (2.9) and (2.10) for p^2 time-like and space-like respectively.

identical to the Pauli-Villars regularizing factor, often adopted in relativistic field theories for the elimination of ultraviolet divergences in perturbation expansions. For the anti-de Sitter group $SO(3, 2)$ (for $P_5 = 1$) we obtained :

$$\Omega(p) = \frac{4M^2}{4M^2 - p^2} . \quad (2.10)$$

The analogous procedure starting from space-time \mathbb{M} will provide convergence factors $\Omega(x)$ which will eliminate infrared divergences. That conformally covariant theories may be free from divergences was also shown by Mack and Todorov [9].

3. The homogeneous space, action of conformal reflections.

In reference [8] it was shown how \mathbb{M}_c and \mathbb{P}_c may be represented as homogeneous spaces of the conformal group $C = L \otimes D \rtimes P^{(4)} \rtimes S^{(4)}$ (where $L, D, P^{(4)}, S^{(4)}$ stand for Lorentz-Dilatation-Poincaré-Special conformal-transformations, respectively).

Precisely:

$$\mathbb{M}_c = \frac{C}{c_1} ; \quad \mathbb{P}_c = \frac{C}{c_2} , \quad (3.1)$$

where $c_1 = L \otimes D \rtimes S^{(4)}$; $c_2 = L \otimes D \rtimes P^{(4)}$. \mathbb{M}_c and \mathbb{P}_c in eq. (3.1) are both isomorphic to $(S^3 \times S^1)/\mathbb{Z}_2$. Furthermore if we represent with I a conformal reflection (a reflection with respect to a plane orthogonal to the 5^{th} or 6^{th} axis) then:

$$I\mathbb{M}_cI^{-1} = \mathbb{P}_c . \quad (3.2)$$

If we now consider the conformal group C inclusive of reflections (represented in $\mathbb{R}^{4,2}$ by $O(4, 2)$), then \mathbb{M}_c and \mathbb{P}_c are two copies of the same homogeneous space of C , transformed in each other by conformal reflections. And then neither \mathbb{M}_c nor \mathbb{P}_c are automorphism spaces for C , but only the two taken together; that is conformally compactified phase space. A conformal reflection I determines a convolution between \mathbb{M}_c and \mathbb{P}_c and then also between \mathbb{M} and \mathbb{P} . This duality, which could be named conformal duality, to distinguished it from the quite different Fourier one, could be of relevance for physics.

The action of I in space-time \mathbb{M} , densely imbedded in \mathbb{M}_c , is well known; for $x_\mu \in \mathbb{M}$:

$$I : x_\mu \rightarrow I(x_\mu) = \pm \frac{x_\mu}{x^2} , \quad (3.3)$$

which, for x_μ space like, is often interpreted as the map of every point, inside a unit sphere S^2 in ordinary space², at a distance x from its centre to a point (on the

²It is interesting to observe that, for $\mathbb{M} = \mathbb{R}^{2,1}$, for which the conformal group is represented by the anti-de Sitter group $O(3, 2)$, the sphere S^2 reduces to a circle S^1 and then eq. (3.4) reminds the Target Space duality in string theory [10], which then might be correlated with conformal duality advocated in this paper.

same ray) at a distance x^{-1} :

$$I : x \rightarrow I(x) = \frac{1}{x} . \quad (3.4)$$

For the physical interpretation x is thought to be dimensionless that is represented by x/L where L is an arbitrary unit at length (and this breaks conformal covariance which is already broken together with Lorentz covariance in (3.4)). With this interpretation a micro world ($x/L \ll 1$) is transformed by I to the macro world ($x/L \gg 1$) in ordinary 3D space.

The corresponding interpretation may be also adopted [8] for the action of I in momentum space $\mathbb{P} = \mathbb{R}^{3,1}$ densely imbedded in \mathbb{P}_c , where, for $k_\mu \in \mathbb{P}$:

$$I : k_\mu \rightarrow I(k_\mu) = \pm \frac{k_\mu}{k^2} \quad (3.5)$$

and correspondingly

$$I : k \rightarrow I(k) = \frac{1}{k} . \quad (3.6)$$

If instead we take into account of (3.2) eq. (3.4) must be interpreted as bringing a point of \mathbb{M} to a point of \mathbb{P} and therefore we do not need to interpret x as dimensionless, we may give it the meaning of a length and then eq. (3.4) becomes:

$$I : x \rightarrow I(x) = \frac{1}{x} = k \in \mathbb{P} \quad (3.7)$$

and similarly eq. (3.6) becomes

$$I : k \rightarrow I(k) = \frac{1}{k} = x \in \mathbb{M} . \quad (3.8)$$

Reminding that C is an automorphism group for phase space and that the physical momentum p is obtained multiplying k by an unit of action $H : p = H \cdot k$, taking together the above equations we arrive to the following interpretation for the action of I in physical phase space:

$$I : \frac{xp}{H} \rightarrow I\left(\frac{xp}{H}\right) = \frac{H}{xp} ; \quad (3.9)$$

which means: in phase space conformal inversion I brings from regions of where the action is $\ll H$ to those where it is $\gg H$; that is from those appropriate for the description of quantum phenomena (in the micro world) to those appropriate for the description of classical phenomena (in the macro world). It could then represent a sort of geometrical prerequisite for the realization of the correspondence principle.

Now, since the conformal inversion I brings also from \mathbb{M}_c to \mathbb{P}_c and vice versa, it would appear that, since obviously \mathbb{M}_c is appropriate for the description of classical mechanics with the geometrical instrument of euclidean geometry, momentum space could be the most appropriate for the description of quantum mechanics, and in this space the most appropriate geometrical instrument for its description seems to be spinor geometry.

4. Spinor representation of quantum mechanics in momentum space.

Fermions are the most elementary constituents of matter. Their properties may be ideally described in the frame of spinor geometry, discovered by E. Cartan [2], which, for what concerns us, may be summarized as follows [11].

Given a real, $2n$ dimensional, vector space V with scalar product g with signature $(k, l); k + l = 2n$, the corresponding Clifford algebra $\mathbb{C}\ell(k, l)$, is central simple and has one, up to equivalence, representation:

$$\gamma : \mathbb{C}\ell(k, l) = \text{End } S_D \quad (4.1)$$

in a complex, 2^n dimensional space S_D of Dirac spinors. If γ_a , obeying $[\gamma_a, \gamma_b]_+ = 2g_{ab}$, are the generators of $\mathbb{C}\ell(k, l)$ and p_a the components of a vector $p \in V$, a Dirac spinor ψ may be defined through the Cartan's equation

$$\sum_{a=1}^{2n} p_a \gamma^a \psi = 0. \quad (4.2)$$

For $\psi \neq 0$, we have that the vector p is null: $p_a p^a = 0$ which implies that the directions of p_a form the compact, projective quadric:

$$P_c = \frac{S^{k-1} \times S^{l-1}}{\mathbb{Z}_2}, \quad (4.3)$$

of which eq. (2.3) is a particular case for the signature $(4, 2)$.

Eq. (4.2) associates to each spinor ψ a totally null plane in V defined by all null, mutually orthogonal vectors $p \in V$ satisfying it. When such a plane has dimension n , that is maximal, the spinor ψ was named simple by E. Cartan (and pure by C. Chevalley).

E. Cartan has further shown how vectors of euclidean geometry in V may be conceived as bilinearly composed of spinors. In fact if we represent the generators γ_a of $\mathbb{C}\ell(k, l)$ as $2^n \times 2^n$ matrices acting on spinor space S , also the transposed matrices γ_a^t , defined by

$$\gamma_a^t = B \gamma_a B^{-1}, \quad (4.4)$$

will generate $\mathbb{C}\ell(k, l)$ and B is uniquely defined since $\mathbb{C}\ell(k, l)$ is simple, and they will act on the dual of S_D . We will have then, for ψ and $\phi \in S_D$ [12]:

$$\psi \otimes B\phi = \frac{1}{2^n} \sum_{k=0}^{2n} \gamma_{\mu_1} \gamma_{\mu_2} \dots \gamma_{\mu_k} B_k^{\mu_1 \mu_2 \dots \mu_k} (\psi, \phi) \quad (4.5)$$

where

$$B_k^{\mu_1 \mu_2 \dots \mu_k} (\psi, \phi) = \langle B\psi, \gamma^{\mu_1} \gamma^{\mu_2} \dots \gamma^{\mu_k} \phi \rangle,$$

in which:

$$1 \leq \mu_1 < \mu_2 \dots \leq 2n.$$

From (4.5) we may then obtain:

$$(\gamma_a \psi \otimes B\phi) \gamma^a \psi = p_a \gamma^a \psi \quad , \quad (4.6)$$

where

$$p_a = \langle B\phi, \gamma_a \psi \rangle. \quad (4.7)$$

Now $p_a p^a = 0$ for either ϕ or ψ simple [12] and then from (4.6) we obtain identically Cartan's eq. (4.2) where the p_a are bilinearly expressed in terms of spinors.

Define now with γ_{2n+1} the volume element of $\mathbb{C}\ell(2n) = \mathbb{C}\ell(k, l)$:

$$\gamma_{2n+1} = i^l \gamma_1 \gamma_2 \dots \gamma_{2n} \quad , \quad (4.8)$$

it anti commutes with all the γ_a , and it may be considered as the $(2n+1)^{th}$ generator of $\mathbb{C}\ell(k+1, l)$ which is a non simple algebra while its even sub algebra $\mathbb{C}\ell_0(2n+1)$ is simple: $\mathbb{C}\ell_0(2n+1) = \text{End}S_P$ and the associated spinors are named Pauli spinors.

The volume element γ_{2n+1} defines the Weyl spinors ψ_{\pm} of opposite helicity of $\mathbb{C}\ell(2n)$ corresponding to each Dirac spinor ψ :

$$\psi_{\pm} = \frac{1}{2} (1 \pm \gamma_{2n+1}) \psi \quad ; \quad \psi_+ + \psi_- = \psi \quad , \quad (4.9)$$

building up the endomorphism spaces of the even subalgebra $\mathbb{C}\ell_0(2n)$ of $\mathbb{C}\ell(2n)$, which is non-simple.

For physical applications we need the vectors p_a given in eq. (4.7) to be real. To this end we introduce the charge-conjugate spinor $\psi_c = C\bar{\psi}$ where $\bar{\psi}$ means ψ complex conjugate and C is defined by $C\gamma_a = \bar{\gamma}_a C$. Then we have [13] that for the signature $(k, l) = (m+1, m-1)$ the vectors:

$$p_a^{\pm} = \langle B\psi_c, \gamma_a (1 \pm \gamma_{2n+1}) \psi \rangle \quad , \quad (4.10)$$

are real (or imaginary) for m even while complex for m odd. That is p_a will be real for the signatures $(3, 1), (5, 3), (7, 4) \dots$ (for $(4, 2)$, that is for twistors, p_a will be complex)³.

It is remarkable that in this way one obtains [13] from Cartan's eq. (4.2), (where also the vectors p_a are conceived as bilinearly composed by spinors) not only the elementary equations of quantum mechanics in first quantization, however in momentum space; including Maxwell's equations (which somehow constitute a bridge between quantum and classical physics), but also those manifesting the so called internal symmetry.

³It may be shown that they are also real for the lorentzian signature $(2n-1, 1)$.

In fact consider the following isomorphisms of algebras:

$$\begin{aligned} \mathbb{C}\ell(2n) & \text{ is isomorphic to } \mathbb{C}\ell_0(2n+1) & - \text{ both simple} \\ \mathbb{C}\ell_0(2n) & \text{ is isomorphic to } \mathbb{C}\ell(2n+1) & - \text{ both non-simple} \end{aligned}$$

which allows to consider a Dirac spinor associated with $\mathbb{C}\ell(2n)$ as a direct sum of Weyl spinor or of Pauli spinor which in turn may be conceived as Dirac spinors of $\mathbb{C}\ell(2n-2)$ and so on. An elementary and historical example is the space-time Dirac spinor, direct sum of right- and left-handed Weyl spinors, which may also be considered as a doublet of Pauli spinors associated with $\mathbb{C}\ell(3)$ (for non relativistic motions).

In this way from the Cartan's eq. (4.2) for Weyl spinors associated with $\mathbb{C}\ell(5,3)$, taking into account of (4.10) the following equation is obtained [13]:

$$(p_\mu \gamma^\mu \cdot \mathbb{I} + \vec{\pi} \cdot \vec{\sigma} \otimes \gamma_5 + m \cdot \mathbb{I}) N = 0 \quad , \quad (4.11)$$

$$\text{where: } \vec{\pi} = \langle \tilde{N}, \vec{\sigma} \otimes \gamma_5 N \rangle; N = \begin{bmatrix} \psi_1 \\ \psi_2 \end{bmatrix}; \tilde{N} = [\tilde{\psi}_1, \tilde{\psi}_2]; \tilde{\psi}_j = \psi_j^\dagger \gamma_0,$$

with ψ_1, ψ_2 - space-time Dirac spinors, and $\vec{\sigma} = (\sigma_1, \sigma_2, \sigma_3)$ Pauli matrices.

Eq. (4.11) is formally identical to the proton-neutron equation interacting with the pseudoscalar isotriplet $\vec{\pi}$ representing the pion, however in momentum space, and the internal isospin symmetry appears as generated by the conformal reflections with respect to the planes orthogonal to the 5th, 6th and 7th axis, and proton-neutron equivalence might represent a natural realization of quaternion algebra.

In fact it is known that a reflection with respect to a plane orthogonal to γ_a is represented in spinor space by $\psi \rightarrow \gamma_a \psi$; and, if γ_a is time-like, it has to be substituted by $i\gamma_a$, if we impose that the square of a reflection equals the identity, which is the case for γ_6 in $\mathbb{C}\ell(5,3)$, which brings to eq. (4.11). Furthermore also the pseudoscalar nature of the pion triplet is uniquely determined from spinor geometry since the representation in which the eighth component spinor N is a doublet of equivalent Dirac spinors, imposes for the gamma matrices to have the form:

$$\Gamma_\mu = \mathbb{I} \otimes \gamma_\mu \quad ; \quad \Gamma_5 = \sigma_1 \otimes \gamma_5 \quad ; \quad \Gamma_6 = i\sigma_2 \otimes \gamma_5 \quad ; \quad \Gamma_7 = \sigma_3 \otimes \gamma_5. \quad (4.12)$$

As we have seen the 8-component spinor N of eq. (4.11) may be also considered as a doublet of Weyl spinors associated with $\mathbb{C}\ell_0(4,2)$, or twistors: $\Psi = \begin{bmatrix} \pi_+ \\ \pi_- \end{bmatrix}$ obeying the Cartan's equation:

$$\left(p^a \tilde{\Gamma}_a + p^8 \cdot \mathbb{I} \right) \Psi = 0 \quad , \quad (4.13)$$

where $\tilde{\Gamma}_a$ have the form:

$$\tilde{\Gamma}_\mu = \sigma_1 \otimes \gamma_\mu \quad ; \quad \tilde{\Gamma}_5 = \sigma_1 \otimes \gamma_5 \quad ; \quad \tilde{\Gamma}_6 = i\sigma_2 \otimes \mathbb{I} \quad ; \quad \tilde{\Gamma}_7 = \sigma_3 \otimes \mathbb{I}. \quad (4.14)$$

Let us now define the 8×8 matrix

$$U = \begin{bmatrix} L & R \\ R & L \end{bmatrix} = U^{-1} ,$$

where $L = \frac{1}{2}(1 + \gamma_5)$; $R = \frac{1}{2}(1 - \gamma_5)$. It is easily seen that

$$U\Gamma_a U^{-1} = \tilde{\Gamma}_a ; \quad UN = \Psi ; \quad U\Psi = N ,$$

from which we have that $(1 \otimes L)N = N_L \equiv (1 \otimes L)\Psi = \Psi_L$.

But then eq.s (4.13) and (4.11) may be summed to give⁴

$$\left(p_\mu + \vec{A}_\mu \cdot \vec{\sigma} \right) \gamma^\mu N_L + BN_R = 0 , \quad (4.15)$$

where $\vec{A}_\mu = \langle \tilde{N}, \vec{\sigma} \otimes \gamma_\mu N \rangle$ and $N_R = (1 \otimes R)N$. If we now suppose that, of the two Dirac spinors ψ_1, ψ_2 , the first represents the electron: e and the second the left handed neutrino: ν_L then eq. (4.15) becomes:

$$\left(p_\mu + \vec{A}_\mu \cdot \vec{\sigma} \right) \gamma^\mu \begin{bmatrix} e_L \\ \nu_L \end{bmatrix} + \alpha e_R = 0 , \quad (4.16)$$

where α is a free parameter.

Eq. (4.16) is the equation of the electroweak model, here derived from eq. s (4.13) and (4.11) both obtained from eq. (4.2) for $\mathbb{C}\ell(5, 3)$. It has been shown [15], [16] that if one considers the triplet e_L, e_L^c, ν_L (or the corresponding 2-component Pauli spinors) to transform with $SU(3)$, the mixing angle Θ results determined such that $\sin^2\Theta = 0.25$. Further details and consequences of these computations will be given elsewhere.

These examples, naturally derived from Cartan's eq. (4.2), representing some of the basic equations of quantum physics in momentum space, in the frame of spinor geometry, may induce to think that the method could be extended also to higher dimensional spinor spaces, e.g. associated with $\mathbb{C}\ell(9, 1)$, in order to explain the multiplicities of elementary fermions (and bosons). In such spinor spaces the problem of dimensional reduction (from 10 to 4, say) of pseudo-euclidean vector spaces through *ad hoc* compactifications, could be avoided [15] [17].

5. Further aspects of conformal duality.

Conformally compactified phase space, conceived as an automorphism space of the extended conformal group, implies the conformal duality between space-time and momentum space which appears as a convolution, determined by conformal reflections. This duality might have two aspects of interest for physics.

⁴The charged vector bosons derived from the 6-vector Z_a , bilinearly generated by twistors π_+ , π_- , which are complex. They identically satisfy the equation $Z_a \gamma^a \pi_+ = 0$ and $\bar{Z}_a \gamma^a \pi_- = 0$, out of which the charged part of eq. (4.15) is obtained.

The first follows from its comparison with Fourier duality which is defined through functions, or physical fields, which may be defined in space time and its Fourier-dual momentum space. As we have seen in general, and in particular in the soluble two dimensional case, for a compact phase space such Fourier dual spaces may be only discrete and finite. As such they might be named the "physical" spaces, to be distinguished from the homogeneous or "mathematical" spaces \mathbb{M}_c and \mathbb{P}_c which are also finite but continuous. The "physical" discrete spaces will be both Fourier and conformally-dual while the "mathematical" spaces will be only conformally dual, and only the former should be the appropriate ones for description of physical phenomena.

The second derives from the possible correlation of conformal duality with the correspondence principle in so far it could be, as shown above, a sort of geometrical prerequisite for the realization of the correspondence principle, once one has found the motivation for identifying the unit of action H introduced in eq. (2.6) with the Planck's constant \hbar (as in the de Broglie equality $p = \hbar \cdot k$). But it could perhaps, in any case, throw some light on some of the still somehow mysterious aspects of the correspondence principle, as we will try to show elsewhere. Furthermore one could expect that some of the geometrical aspects, that is of the topological and symmetry properties which are common to both \mathbb{M}_c and \mathbb{P}_c could be manifested by both classical and quantum physical systems independently (and above) of the correlation, specifically predicated by the correspondence principle (identification of wave functions with classical orbits for high quantum numbers). One of them is the $SO(4)$ symmetry which could be identified as the maximal compact subgroup of $SO(4,2)$ and manifested by the presence of its isometry sphere S^3 in both \mathbb{M}_c and in its conformally dual \mathbb{P}_c ; and then to be expected in both classical and quantum mechanical stationary (non relativistic) systems in ordinary- and momentum-space respectively. These systems exist they are the planetary motions in space-time and the H-atom in momentum space. In fact it is remarkable that the $SO(4)$ symmetry of the H-atom was discovered by V. Fock in the S^3 compactification of momentum space [18], which suggests that this $SO(4)$ symmetry might be a consequence of conformal duality, rather than being "accidental", as it was named by W. Pauli when discovered.

With R. Raczka [19] we have also conjectured an eigenvibration of the S^3 sphere of the Robertson Walker universe represented by

$$M_{RW} = S^3 \times R^1, \quad (5.1)$$

(which is often considered as a natural realization of \mathbb{M}_c given by eq. (2.1) where R^1 is the infinite covering of S^1), in order to explain a remarkable regularity in the distribution of distant galaxies [20] (more than eleven peaks equally spaced by about $4 \cdot 10^8$ light years, in the direction of the North and South galactic poles). We have shown [21] how the astronomical data are well represented by the most symmetric

spherical harmonic of S^3 :

$$Y_{n,0,0} = k_n \frac{\sin (n + 1) \chi}{\sin \chi} \quad (5.2)$$

where χ is the geodesic distance from center of the eigenvibration.

If further astronomical observations will confirm this model it would represent a remarkable test of conformal duality since eq. (5.2) is exactly the eigenfunction of the stationary S-states of the H-atoms found by V. Fock precisely in S^3 compactification of momentum space (eq. (26) of ref. [18]). Then the Universe and the H-atom would constitute an example of realization of conformal duality, representing two conformally dual systems having the same eigenfunction : the first in ordinary space and the second in the conformally dual momentum space.

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