

On Fermions in Compact Momentum Spaces Bilinearly Constructed with Pure Spinors

Paolo Budinich*

Abstract

It is shown how the old Cartan's conjecture on the fundamental role of the geometry of simple (or pure) spinors, as bilinearly underlying euclidean geometry, may be extended also to quantum mechanics of fermions (in first quantization), however in compact momentum spaces, bilinearly constructed with spinors, with signatures unambiguously resulting from the construction, and locally conceived as both Fourier- and conformally-dual to space-time, where classical mechanics is traditionally described. In this construction most of the elementary equations of motion of fermion-physics, usually postulated ad hoc, are naturally obtained from Cartan's equation defining spinors. We start from two-component spinors obeying Weyl equation for massless neutrinos (from which in turn Maxwell equations are derived) following with eight-component spinor-equations representing charged-neutral fermion doublets presenting $SU(2) \otimes U(1)$ internal symmetry, including the skeleton of the electroweak model, up to sixteen component Majorana-Weyl spinors associated with the real Clifford algebra $\mathbb{C}\ell(1, 9)$, where, because of the known periodicity theorem, the construction naturally ends. $\mathbb{C}\ell(1, 9)$ may be formulated in terms of the octonion division algebra, at the origin of $SU(3)$ internal symmetry, and which seems appropriate to furnish a natural explanation for several of the observed properties of baryon- and lepton-physics.

In this approach the extra dimensions beyond 4 appear as interaction terms in the equations of motion of the fermion multiplet; more precisely the directions from 5th to 8th correspond to electric, weak and isospin interactions ($SU(2) \otimes U(1)$), while those from 8th to 10th

*International School for Advanced Studies, Trieste 34014, Italy

to strong ones ($SU(3)$). Furthermore, dimensional reduction in momentum space, which is compact, may be simply identified with “decoupling”. In fact it is generated by operators which are extensions of the familiar chiral projectors in spinor space, and they then naturally reduce both the spinor space of fermion multiplets (reduced by a factor 1/2) and the corresponding interaction terms, of which two are eliminated, corresponding to two extra dimensions in momentum space, for which then there seems to be no need of the corresponding configuration-space. Only four dimensional space-time is needed – for the equations of motion and for the local fields – and also naturally generated by four-momenta as Poincaré translations.

This spinor approach could be compatible with string theories and even explain their origin, since also strings may be bilinearly obtained from simple (or pure) spinors through sums; that is integrals of null vectors – which are generalizations of the old Enneper-Weierstrass parametrization of minimal surfaces – in the case of real Clifford algebras, like $\mathbb{C}\ell(1, 9)$, admitting Majorana-Weyl spinors.

1 INTRODUCTION

In the study of the elementary constituents of matter, the discovery of the existence of fermion- and boson-multiplets, which could be labeled according to the representations of certain groups ($SU(2)$, $SU(3)$...), named internal symmetry groups, has brought to the conjecture of the existence of higher dimensional space-times, in which to imbed ordinary space-time. The non-observable dimensions (> 4) were then eliminated with appropriate methods (e.g. Kaluza Klein), confining them in compact manifolds of invisible size.

Here we propose a more economical “constructive” approach which simply consists in representing the observed fermions, by Pauli, Weyl or Dirac spinors to be embedded in higher dimensional spinor spaces and consequently the same for their endomorphism- Clifford algebras, and, at each step of this construction, which unambiguously defines the signatures of the higher dimensional Clifford algebras, observe which are the equations of motion whose geometry is contained in the Cartan’s equation defining spinors, if interpreted in momentum space.

The main motivation of this approach is not only economy; that is to introduce, in the attempts of theoretical explanations, only actually observed geometrical objects that is spinors, representing fermions,

(the bosons may be notoriously expressed, bilinearly, in terms of spinors), but rather to literally adhere to the hypothesis of E. Cartan [1] who conjectured that the fundamental geometry appropriate for the description and understanding of elementary natural phenomena is spinor geometry, more precisely the geometry of simple spinors, later named pure by C. Chevalley [2], rather than the one of euclidean vectors which may be constructed bilinearly from spinors.

After a short review on the properties of simple or pure spinors in Chapter 2, we will start by considering the elementary case of two component Pauli spinors associated with 3-dimensional euclidean space and show how we may obtain from their Clifford algebra both the signature of space-time and Weyl and Maxwell's equations (Chapter 3), in momentum space.

We will then study in Chapter 4 the problem of imbedding spinor spaces and null vector spaces in higher dimensional ones, and we will show how it may be easily solved, in the case of simple or pure spinors, with the use of two propositions. In Chapter 5 we will deal with four component spinors and with the Dirac equation for massive fermions and the Cartan's equation for Weyl spinors or twistors.

We will show then that the minimal Clifford algebra to contain simple Weyl spinors isomorphic to doublets of Dirac spinors is $\mathbb{C}\ell(7, 1)$ or $\mathbb{C}\ell(1, 7)$ associated with eight dimensional vector spaces and how from this both the equation for the nucleon doublet interacting with the pseudoscalar pion isovector may be naturally obtained, in which the pion is bilinearly expressed in terms of the spinors of the nucleon doublet (section 6.1), and also the geometrical skeleton of the Salam-Weinberg model of electroweak interactions (section 6.2); all this in momentum space.

We will then show, in section 6.3, how the equations for fermion doublets naturally present an $U(1)$ symmetry, for one of the fermions of the doublet, interpretable as charge, which may be correlated with non equivalent spinor structures in conformal-like theories and furnish then a geometrical explanation of the existence of charged – neutral fermion doublets like electron-neutrino, proton-neutron, etc.

These equations, which historically have been proposed ad hoc for the description of the mentioned phenomena, here seem to have a unique geometrical origin in spinor geometry. They are however to be interpreted in momentum space, thus supporting the conjecture, formulated some time ago, that it is momentum space the appropriate space for the geometrical formulation of quantum mechanics (in first

quantization) [3]. And if we follow the suggestion of Cartan to privilege simple or pure spinors, that is maximal totally null planes laying in Klein quadrics, these momentum spaces are compact, thus a priori eliminating the severe difficulty of ultraviolet divergences.

The internal symmetry ($SU(2)$ in the mentioned case) that thus arises appears to be generated, in flat spaces, by discrete groups of reflection operators, of the conformal group. Through reflections the heuristic approach above can be derived from the hypothesis of the fundamental role of conformal covariance which could impose the compact structure of phase space where space-time and momentum space appear as conjugate with respect to conformal reflections (Chapter 7).

16-component spinors will be studied in Chapter 8 where a new $U(1)$ -charge appears for one of the fermion doublets, which, if interpreted as strong charge, suggests the interpretation of the multiplet as baryon-lepton quadruplet.

In Chapter 9 we will study 32-component spinors associated with the algebra $\mathbb{C}\ell(1, 9)$ where the "construction" finds a natural end, since after that, due to the periodicity theorem, the sequence will be repeated.

In Chapter 10 we will discuss the problem of dimensional reduction from $\mathbb{C}\ell(1, 9)$ which will simply consist in reversing the steps of the "construction", with the use of projector operators in spinor space. There are 3+1 of them corresponding to the dimensions of quaternion-space which, as shown in Chapter 11, section 11.3, might explain the geometrical origin of the "families". In Chapter 12 the octonion formalism is introduced in the frame of $\mathbb{C}\ell(1, 9)$ to show that it may give rise to several $SU(3)$ internal symmetry subgroups of G_2 in the equations of the baryon multiplet. Further geometrical aspects are mentioned in Chapter 13, among these simplicity constraints in section 13.1.

The present approach, rather than alternative to the now prevailing approach based on strings, could be compatible with it, since strings may be formulated as integrals of null-vectors which are notoriously the characteristic transition elements (bilinear) from spinor to euclidean geometry. Spinors then could be at the origin of strings insofar these could merely represent the intermediate stage between spinor- and euclidean-geometry as mentioned in section 13.2.

2 SIMPLE- OR PURE-SPINOR GEOMETRY

The geometry of simple or pure spinors was discovered by É. Cartan. He formulated the basic axioms and theorems which, for what concerns us, may be summarized [4] as follows.

Given a $2n$ -dimensional complex space $W = \mathbb{C}^{2n}$ and the corresponding central simple Clifford algebra $\mathbb{C}\ell(2n)$ with generators γ_a , ($a = 1, \dots, 2n$), obeying $\{\gamma_a, \gamma_b\} = (-1)^{a+1}2\delta_{ab}$, a spinor ϕ is a vector of the complex 2^n dimensional representation space S of $\mathbb{C}\ell(2n) = \text{End}S$, defined by

$$z_a \gamma^a \phi = 0 \quad (2.1)$$

where z_a are the orthonormal components of a vector $z \in W$. This vector is null since, for $\phi \neq 0$, eq. (2.1) defines the Klein quadric $Q \subset W$:

$$Q: \quad z_a z^a = 0 \quad (2.2)$$

For fixed ϕ , all $z \in W$ satisfying (2.1) and (2.2) define a totally null plane $T(\phi) \subset W$, whose vectors are both null and mutually orthogonal.

Let $\gamma_{2n+1} = \gamma_1 \cdot \gamma_2 \cdots \gamma_{2n}$ represent the volume element (normalized to 1) of $\mathbb{C}\ell(2n)$, then the spinors ϕ_+ , ϕ_- obeying

$$\gamma_{2n+1} \phi_{\pm} = \pm \phi_{\pm} \quad (2.3)$$

are named Weyl spinors and for them the defining equation (2.1) becomes

$$z_a \gamma^a (1 \pm \gamma_{2n+1}) \phi_{\pm} = 0 \quad (2.4)$$

A Weyl spinor ϕ_+ or ϕ_- is named simple or pure if the associated totally null plane $T(\phi_{\pm})$ is n -dimensional, that is, maximal and we will write $T(\phi_{\pm}) = M(\phi_{\pm})$. For $n \leq 3$ all Weyl spinors ϕ_+ , ϕ_- are simple.

The 2^{n-1} dimensional spaces S_+ and S_- of Weyl spinors are endomorphism spaces of the even Clifford subalgebra $\mathbb{C}\ell_0(2n)$ of $\mathbb{C}\ell(2n)$ that is:

$$\mathbb{C}\ell_0(2n) = \text{End}S_{\pm}. \quad (2.5)$$

We have further:

$$\phi = \phi_+ \oplus \phi_- , \quad S = S_+ \oplus S_- . \quad (2.6)$$

and $\mathbb{C}l(2n) = 2\mathbb{C}l_0(2n)$.

These definitions may easily be extended also to odd dimensional spaces, in fact, since the volume element γ_{2n+1} , for every γ_a , obey to: $\{\gamma_a, \gamma_{2n+1}\} = 2\delta_{a,2n+1}$, $1 \leq a \leq 2n$ we have that $\gamma_1, \gamma_2, \dots, \gamma_{2n}, \gamma_{2n+1}$ generate the Clifford algebra $\mathbb{C}l(2n+1)$ of the complex vector space $W = \mathbb{C}^{2n+1}$ and there is the isomorphism [4]:

$$\mathbb{C}l_0(2n+1) \simeq \mathbb{C}l(2n) \quad (2.7)$$

both being simple. The corresponding 2^n component spinors are called Pauli spinors: ϕ_P , for $\mathbb{C}l_0(2n+1)$ and Dirac spinors: ϕ_D for $\mathbb{C}l(2n)$.

$\mathbb{C}l(2n+1)$ instead is not simple and there is the isomorphism [4]:

$$\mathbb{C}l(2n+1) \simeq \mathbb{C}l_0(2n+2) \quad (2.7')$$

For embedding spinors in higher dimensional spinors we may then use eqs. (2.5), (2.6), (2.7) and (2.7'):

$$\begin{aligned} \mathbb{C}l(2n) &\simeq \mathbb{C}l_0(2n+1) \hookrightarrow \mathbb{C}l(2n+1) \simeq \\ &\simeq \mathbb{C}l_0(2n+2) \hookrightarrow \mathbb{C}l(2n+2) \dots \end{aligned}$$

corresponding to the spinor embeddings:

$$\psi_D \simeq \psi_P \hookrightarrow \psi_P \oplus \psi_P \simeq \psi_W^+ \oplus \psi_W^- = \Psi = \psi_D \oplus \psi_D$$

which means that a 2^n components Dirac spinors may be considered equivalent to a doublet of 2^{n-1} components Weyl, Pauli or Dirac spinors. For simple spinors, these embeddings will correspond to embeddings of Klein quadric in higher dimensional ones which, when restricted to the real and projective quadrics, will correspond to the embedding of compact manifolds in higher dimensional compact manifolds.

The vectors (and tensors) of the manifolds, to be interpreted as vector spaces of physics, will result bilinearly expressed in terms of spinors, therefore we need to define inner products in spinor spaces. This is easily done through the main anti-automorphism of the simple Clifford algebras $\mathbb{C}l(2n)$ or $\mathbb{C}l_0(2n+1)$ which defines an isomorphism $B : S \rightarrow S^*$ where S^* is the dual spinor space of S such that

$$B\gamma_a = \gamma_a^t B \quad \text{and} \quad B\phi = \phi^t B \in S^* \quad (2.8)$$

where γ_a^t and ϕ^t mean γ_a and ϕ transposed, respectively.

If ψ is another spinor of S we have then the invariant (for the *Pin* group) scalar product:

$$\langle \phi^*, \psi \rangle = \langle B\phi, \psi \rangle = \phi^t B\psi.$$

In the case of real vector spaces, of interest for physics, we will also need to define the conjugation operator C such that

$$C\gamma_a = \bar{\gamma}_a C \quad \text{and} \quad \phi^c = C\bar{\phi} \quad (2.9)$$

where $\bar{\gamma}_a$ and $\bar{\phi}$ mean γ_a and ϕ complex conjugate, respectively.

Another useful definition of simple spinors may be obtained through the formula

$$\phi \otimes B\psi = \sum_{j=0}^n F_j \quad (2.10)$$

where $\phi, \psi \in S$ are spinors of $\mathbb{C}\ell(2n) = \text{End}S$ and

$$F_j = [\gamma_{a_1}\gamma_{a_2} \cdots \gamma_{a_j}] T^{a_1 a_2 \cdots a_j} \quad (2.10')$$

where the γ_a products are antisymmetrized and $T^{a_1 a_2 \cdots a_j}$ is an anti-symmetric j -tensor of \mathbb{C}^{2n} , which can be expressed bilinearly in terms of the spinors ϕ and ψ as follows:

$$T_{a_1 a_2 \cdots a_j} = \frac{1}{2^n} \langle B\psi, [\gamma_{a_1}\gamma_{a_2} \cdots \gamma_{a_j}] \phi \rangle \quad (2.11)$$

Setting now $\psi = \phi$, in eq. (2.10) we have that ϕ is simple if and only if

$$F_0 = 0, \quad F_1 = 0, \quad F_2 = 0, \dots, F_{n-1} = 0 \quad (2.12)$$

while $F_n \neq 0$ and eq. (2.10) becomes:

$$\phi \otimes B\phi = F_n \quad (2.13)$$

and the n -tensor F_n represents the maximal totally null plane of W equivalent, up to a sign, to the simple spinor ϕ . Equations (2.12) represent then the constraint equations for a spinor ϕ to be a simple or pure spinor associated with W .

Eq. (2.13) represents then the correspondence of simple or pure spinor directions with maximal totally null planes sometimes called “the Cartan map”.

The equivalence of this definition with the one deriving from eq. (2.1), given by Cartan, is easily obtained if we multiply eq. (2.10) on the left by γ_a and on the right by $\gamma_a\phi$ and sum over a , obtaining

$$\gamma_a\phi \otimes B\psi\gamma^a\phi = z_a\gamma^a\phi \quad (2.14)$$

where

$$z_a = \langle B\psi, \gamma_a\phi \rangle$$

which, provided ϕ is simple or pure, for arbitrary ψ , satisfy

$$z_a\gamma^a\phi = 0.$$

and z_a are the components of a null vector of W , belonging to F_n .

This result may be formulated as follows:

Proposition 1. Given a complex space $W = \mathbb{C}^{2n}$ with its Clifford algebra $\mathbb{C}\ell(2n)$, with generators γ_a , let ψ and ϕ represent two spinors of the endomorphism spinor-space of $\mathbb{C}\ell(2n)$ and of its even subalgebra $\mathbb{C}\ell_0(2n)$, respectively. Then, the vector $z \in W$, with components:

$$z_a = \langle B\psi, \gamma_a\phi \rangle; \quad a = 1, 2, \dots, 2n \quad (2.15)$$

is null if and only if ϕ is a simple or pure spinor. For fixed ϕ simple or pure, and arbitrary ψ all vectors of the maximal, totally null plane $M(\phi)$ in W , are so obtained. The proof is given in reference [6].

The above formalism may be easily restricted to the real, simply substituting the complex space $W = \mathbb{C}^{2n}$ with the real neutral pseudo-euclidean space $V = \mathbb{R}^{n,n}$. The corresponding Clifford algebra $\mathbb{C}\ell(n, n)$ is generated by the generators γ_a , satisfying:

$$\{\gamma_a, \gamma_b\} = (-1)^{a+1} 2\delta_{ab} \quad (2.16)$$

and, in the previous computations the complex components z_a of the vector $z \in W$ have to be substituted by real ones p_a of $p \in V$, and in eqs. (2.10), (2.11), (2.14) and (2), $B\phi$ and $B\psi$ have to be substituted by $B\phi^c$ and $B\psi^c$, respectively.

The same formalism may be also extended [5] to real pseudo-euclidean spaces $\mathbb{M} = \mathbb{R}^{n+1, n-1}$ which, for $n = 2$, identifies with Minkowski space-time and, for $n > 2$, represents its conformal extensions, as well as of its Fourier dual momentum space. In this case z_a are real (or imaginary) only for n even, that is for $\mathbb{C}\ell(3, 1)$, $\mathbb{C}\ell(5, 3)$,

$\mathbb{C}\ell(7, 5)$ etc. (while they are complex for n odd, that is for $\mathbb{C}\ell(4, 2)$, $\mathbb{C}\ell(6, 4)$ etc.)

In fact in this case, of interest for physics we have:

Corollary 1. Let $\mathbb{C}\ell(n+1, n-1) = \text{End}S$, the vectors with components

$$z_a^\pm = \langle B\psi^c, \gamma_a\phi^\pm \rangle \quad (2.17)$$

where ψ is an arbitrary spinor of S , are null iff ϕ^\pm are simple Weyl spinors of S . For $\psi = \phi^\pm$ and n even $z_a^\pm = p_a^\pm$ are real (or imaginary), for n odd z_a^\pm are complex such that $\bar{z}_a^+ = \pm z_a^-$.

It is seen that eq. (2.17) is a particular case of eq. (2.15). In reference [6], it is also shown that real vectors are obtained in the case of Lorentzian signature: $\mathbb{C}\ell(2n-1, 1)$.

For n even then, p_a given by (2.17) define the Klein quadric Q given by eq. (2.2) where z_a is substituted by p_a , in $V = \mathbb{R}^{n+1, n-1}$.

It is easily seen that the corresponding projective quadric PQ define the following compact manifold PQ in $\mathbb{R}^{n+1, n-1}$:

$$PQ = \frac{S^n \times S^{n-2}}{\mathbb{Z}_2}. \quad (2.18)$$

where $\mathbb{Z}_2 = [+1, -1]$ means that the antipodal points of PQ are to be identified.

We have now listed the main geometrical instruments of simple or pure spinor geometry useful in order to proceed with the program of imbedding spinor spaces in higher dimensional spinor spaces and to explore which geometrical objects of possible physical meaning we might obtain through this procedure.

We will start with the simplest non trivial case of two component spinors and then proceed to higher ones and we will see that we will obtain, together with Maxwell's equations also most of the elementary equations of fermion physics in momentum space known to us, and nothing else. In other words it appears that every one of the geometrical structures potentially contained in simple or pure spinor geometry are realized in some elementary law of physics supporting thus the Cartan's conjecture of the fundamental role of simple spinor geometry not only for euclidean geometry but also for physics of fermions the most elementary constituents of matter, whose equations of motion already contain some of the geometrical elements of quantum physics (in first quantization: the non-relativistic limit of Dirac equation is Schrödinger equation, up to the definition of the Plank's constant).

3 FROM TWO TO FOUR COMPONENT SPINORS.

3.1 THE SIGNATURE OF SPACE- TIME AND WEYL EQUATIONS FOR MASSLESS NEUTRINOS

Let us start from $W = \mathbb{C}^3$, the generators $\sigma_1, \sigma_2, \sigma_3$ of its Clifford algebra $\mathbb{C}\ell(3) = \text{End}S$, are Pauli matrices and its Pauli spinors $\varphi = \begin{pmatrix} \varphi_0 \\ \varphi_1 \end{pmatrix} \in S$ are simple.¹ In fact we have $B = -i\sigma_2 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ and eq. (2.13) becomes:

$$\begin{pmatrix} \varphi_0\varphi_1 & -\varphi_0\varphi_0 \\ \varphi_1\varphi_1 & -\varphi_0\varphi_1 \end{pmatrix} \equiv \varphi \otimes B\varphi = z_j\sigma_j \equiv \begin{pmatrix} z_3 & z_1 - iz_2 \\ z_1 + iz_2 & -z_3 \end{pmatrix} \quad (3.1)$$

since $F_0 = \langle B\varphi, \varphi \rangle \equiv 0$. Furthermore, since $\frac{1}{2}\langle B\varphi, \sigma_j\varphi \rangle = z_j$ (compare the two matrices), following from eq. (3.1), equation:

$$z_1^2 + z_2^2 + z_3^2 = 0 \quad (3.2)$$

is identically satisfied as may be immediately seen from the determinant of the matrices in eq. (3.1). Also the Cartan's equation:

$$z_j\sigma^j\varphi = 0 \quad (3.3)$$

is identically satisfied, (as may be immediately seen if we act with the first term of eq. (3.1) on φ).

If $\psi \in S$ is another spinor we have, from eq.(2.10):

$$\begin{pmatrix} \varphi_0\psi_1 & -\varphi_0\psi_0 \\ \varphi_1\psi_1 & -\varphi_1\psi_0 \end{pmatrix} \equiv \varphi \otimes B\psi = z_0 + z_j\sigma_j \equiv \begin{pmatrix} z_0 + z_3 & z_1 - iz_2 \\ z_1 + iz_2 & z_0 - z_3 \end{pmatrix}, \quad (3.4)$$

¹Because of the isomorphism $\mathbb{C}\ell_0(3) \simeq \mathbb{C}\ell(2)$ the two components spinors φ may be also conceived as a Dirac spinor associated with $V = \mathbb{C}^2$, which however may not be simple: in fact $\langle B\varphi, \sigma_3\varphi \rangle \neq 0$. In general only Pauli or Weyl spinors may be simple, Dirac ones may not (unless they are conceived as isomorphic to Pauli or Weyl spinors in force of the isomorphisms (2.7), (2.7')). This is presumably the reason why in the introduction of spinor geometry Cartan preferred odd dimensional spaces, in particular \mathbb{C}^3 , to even dimensional ones.

and $z_0 = \frac{1}{2}\langle B\psi, \varphi \rangle$, $z_j = \frac{1}{2}\langle B\psi, \sigma_j \varphi \rangle$ deriving from it, satisfy identically the equation (as may be immediately seen from the determinant of the matrices):

$$z_1^2 + z_2^2 + z_3^2 - z_0^2 = 0 \quad (3.5)$$

which uniquely determines the signature of Minkowski space-time. In fact the above may be easily restricted to the real by substituting $B\psi$ with $B\phi^c = \phi^\dagger$ by which z_0, z_j become p_0, p_j real:

$$p_0 = \frac{1}{2}\langle \varphi^\dagger, \varphi \rangle; \quad p_j = \frac{1}{2}\langle \varphi^\dagger, \sigma_j \varphi \rangle \quad (3.6)$$

satisfying identically to:

$$p_1^2 + p_2^2 + p_3^2 - p_0^2 = 0 \quad (3.7)$$

a null vector or light ray of Minkowski space $\mathbb{R}^{3,1}$. In this case then exploiting the Clifford algebras isomorphism

$$\mathbb{C}\ell(3) \simeq \mathbb{C}\ell_0(3, 1) \quad (3.8)$$

φ may be interpreted as a simple Weyl spinor of S where $\mathbb{C}\ell_0(3, 1) = \text{End}S_\pm$; and there are two of them: φ_+, φ_- satisfying the equations

$$(\vec{p} \cdot \vec{\sigma} + p_0) \varphi_+ = 0 \quad (\vec{p} \cdot \vec{\sigma} - p_0) \varphi_- = 0 \quad (3.9)$$

These equations, and the ones in the following sections, may be interpreted as field equations if we define the Clifford algebras as fibers over momentum basis. For $p_0 = 0$ eqs. (3.9) identify with eq. (3.3) and, in agreement with the isomorphism (2.7) the Weyl spinors φ_\pm associated with $\mathbb{C}\ell_0(3, 1)$ identify with the Pauli spinor φ associated with $\mathbb{C}\ell_0(3)$, apt to represent a fermion spin in non relativistic phenomena.

Eqs.(3.9) may be expressed as a single equation for the four component Dirac space-time spinor $\psi = \varphi_+ \oplus \varphi_-$. In fact indicating with

$$\gamma_\mu = \{\gamma_0; \gamma_1, \gamma_2, \gamma_3\} := \{-i\sigma_2 \otimes 1; \sigma_1 \otimes \sigma_j\}; \quad j = 1, 2, 3$$

the generators of $\mathbb{C}\ell(3, 1)$ and with $\gamma_5 = -i\gamma_0\gamma_1\gamma_2\gamma_3 = \sigma_3 \otimes 1$ the volume element, we may write (3.9) in the form

$$p_\mu^\pm \gamma^\mu (1 \pm \gamma_5) \psi = 0 \quad (3.9')$$

where $\frac{1}{2}(1 \pm \gamma_5)$ represent the chiral projectors. These equations are the Weyl equations for massless neutrinos in momentum space. The null vectors p_μ^\pm may be expressed in the form:

$$p_\mu^\pm = \frac{1}{2} \tilde{\psi} \gamma_\mu (1 \pm \gamma_5) \psi \quad (3.10)$$

where $\tilde{\psi} = \psi^\dagger \gamma_0$.

The space-time Weyl spinors

$$\varphi_\pm = \frac{1}{2} (1 \pm \gamma_5) \psi \quad (3.11)$$

represent massless neutrinos, eigenstates of γ_5 .

For future use let us now compute the components of their intrinsic angular momentum (in units of $\frac{\hbar}{2}$) with respect to the z -axis in space; which is represented by $\frac{1}{2}\sigma_3$, therefore, since

$$\frac{1}{2}(\gamma_5 + 1 \otimes \sigma_3) = \begin{pmatrix} +1 & & & \\ & 0 & & \\ & & 0 & \\ & & & -1 \end{pmatrix}, \quad (3.11')$$

it will be $\pm \frac{\hbar}{2}$ for φ_\pm .

From eq. (3.9') we may also easily obtain Maxwell's equations in momentum space.

3.2 MAXWELL'S EQUATIONS

The Weyl space-time spinor φ_+ is simple and therefore with it eq. (2.13) becomes:

$$\varphi_+ \otimes B\varphi_+ = \frac{1}{2} F_+^{\mu\nu} [\gamma_\mu, \gamma_\nu] (1 + \gamma_5) \quad (3.12)$$

where the antisymmetric tensor $F_+^{\mu\nu}$ may be expressed bilinearly in terms of spinors through eq. (2.11):

$$F_+^{\mu\nu} = \tilde{\psi} [\gamma^\mu, \gamma^\nu] (1 + \gamma_5) \psi, \quad (3.13)$$

and, as already observed by É. Cartan [1], it has the geometrical properties of the electromagnetic tensor. Now from Weyl equation $p_\rho \gamma^\rho \varphi_+ = 0$ we obtain from (3.12):

$$p_\rho \gamma^\rho F_+^{\mu\nu} [\gamma_\mu, \gamma_\nu] (1 + \gamma_5) = 0 \quad (3.14)$$

and since

$$\gamma_\rho \gamma_\mu = g_{\rho\mu} + \frac{1}{2} [\gamma_\rho, \gamma_\mu]$$

it becomes

$$p_\rho F_+^{\rho\nu} \gamma_\nu = 0 \quad (3.15)$$

which is the image in $\mathbb{C}\ell(3, 1)$ of the Maxwell's equation (in vacuum) for the self dual electromagnetic tensor in momentum space $\mathbb{R}^{3,1}$:

$$p_\rho F_+^{\rho\nu} = 0. \quad (3.16)$$

It is easily seen that if we start from the left-handed Weyl spinor φ_- we obtain the other equation:

$$\varepsilon^{\lambda\rho\mu\nu} p_\rho F_{\mu\nu}^- = 0 \quad (3.17)$$

Also the Maxwell's equations in presence of electromagnetic sources may be easily obtained from simple spinor geometry [7].

Observe that from eq. (3.13) it appears that the electromagnetic tensor $F_{\mu\nu}$ is bilinearly expressed in terms of the Weyl spinors φ_+ and φ_- obeying the equation of motion eq. (3.9') of massless neutrinos. This however does not imply that in the quantized theory the photon must be conceived as a bound state of neutrinos. In fact it is known that the neutrino theory of light, while violating both gauge invariance and statistics, is unacceptable [8].

4 IMBEDDING SPINOR SPACES AND NULL-VECTOR SPACES IN HIGHER DIMENSIONAL ONES

We have seen how two component spinors $\varphi = \begin{pmatrix} \varphi_0 \\ \varphi_1 \end{pmatrix}$ may be interpreted as Pauli spinors of $\mathbb{C}\ell(3)$ or, as Weyl spinors of $\mathbb{C}\ell_0(3, 1)$. For these the Cartan's equations identify with Weyl equations for massless fermions or neutrinos and give rise to Maxwell's equations, both in momentum space. In order to proceed with our constructive program we have now first to imbed two-component spinor spaces in four component ones and then in eight component ones and so on; and

at each step analyze which are the corresponding Clifford algebras together with Cartan's equations to see if they may represent some further elementary laws of physical phenomena.

At first sight this program might appear of difficult realization since null vectors are "squares" of spinors and it might appear difficult to obtain from sums of spinors sums of vectors necessary for the imbedding of null spaces in higher dimensional ones. Instead the program is simple if we adopt Proposition 1 and exploit the properties of simple or pure spinors.

In fact let us consider $\mathbb{C}\ell(2n) = \text{End } S$ and $\psi, \phi \in S$, then, for ψ arbitrary and $\varphi_{\pm} = 1/2(1 \pm \gamma_{2n+1})\phi$ simple, the vectors $z^{\pm} \in W$ with components

$$z_a^{\pm} = \langle B\psi, \gamma_a(1 \pm \gamma_{2n+1})\phi \rangle, \quad a = 1, 2, \dots, 2n \quad (4.1)$$

are of the form (2.15) and therefore are null:

$$z_a^{\pm} z_{\pm}^a = 0. \quad (4.2)$$

Let us sum them and we obtain:

$$z_a^+ + z_a^- = Z_a = \langle B\psi, \gamma_a\phi \rangle, \quad a = 1, 2, \dots, 2n, \quad (4.3)$$

where $\phi = \varphi_+ \oplus \varphi_- \in S$, while $\varphi_{\pm} \in S_{\pm}$ where $\mathbb{C}\ell_0(2n) = \text{End } S_{\pm}$ and therefore:

$$S = S_+ \oplus S_-. \quad (4.4)$$

This means that starting from two simple spinor spaces we operated the most obvious operation: their direct sum spanned by spinors of double dimension. Let us now examine the corresponding operation in the vector space W ; it is represented by eq.(4.3), defining the Z_a , they are, in general, the components of a non null vector of \mathbb{C}^{2n} :

$$Z_a Z^a \neq 0. \quad (4.5)$$

However, $\phi = \varphi_+ \oplus \varphi_-$ is a 2^n component spinor and as such it could represent a simple spinor of $\mathbb{C}\ell_0(2n+2)$. Let us in fact take for the generators Γ_A ($A = 1, 2, \dots, 2n+2$) of $\mathbb{C}\ell(2n+2)$ the Cartan basis (by which we mean that the corresponding Dirac spinor is a direct sum of Weyl spinors):

$$\Gamma_A: \Gamma_a = \begin{pmatrix} 0 & \gamma_a \\ \gamma_a & 0 \end{pmatrix}, \quad \Gamma_{2n+1} = \begin{pmatrix} 0 & \gamma_{2n+1} \\ \gamma_{2n+1} & 0 \end{pmatrix} \\ \Gamma_{2n+2} = \begin{pmatrix} 0 & 1_n \\ -1_n & 0 \end{pmatrix} \quad (4.6)$$

and the volume element:

$$\Gamma_{2n+3} = \Gamma_1 \Gamma_2 \cdots \Gamma_{2n+2} = \begin{pmatrix} 1_n & 0 \\ 0 & -1_n \end{pmatrix} \quad (4.7)$$

Then if Φ is a 2^{n+1} component Dirac spinor of $\mathbb{C}\ell(2n+2)$ it has, in the Cartan basis, the form:

$$\Phi = \begin{pmatrix} \phi'_+ \\ \phi'_- \end{pmatrix} \quad (4.8)$$

where the 2^n component Weyl spinors are given by

$$\phi'_\pm = \frac{1}{2} (1 \pm \Gamma_{2n+3}) \Phi \quad (4.9)$$

Suppose they are simple, then the vectors $Z_\pm \in \mathbb{C}^{2n+2}$ with components:

$$Z_A^\pm = \langle B\Psi \Gamma_A (1 \pm \Gamma_{2n+3}) \Phi \rangle, \quad (4.10)$$

with Ψ arbitrary spinor, are of the form (2.15) and will be null:

$$Z_A^\pm Z^A_\pm = 0; \quad A = 1, 2, \dots, 2n+2 \quad (4.11)$$

It is easy to see that in the Cartan-basis (4.6) Z_A^\pm are expressed by

$$\begin{aligned} Z_a^\pm &= \langle B\psi, \gamma_a \phi'_\pm \rangle; \quad a = 1, 2, \dots, 2n \\ Z_{2n+1}^\pm &= \langle B\psi, \gamma_{2n+1} \phi'_\pm \rangle \\ Z_{2n+2}^\pm &= \langle B\psi, \pm 1_n \phi'_\pm \rangle \end{aligned} \quad (4.12)$$

If we now identify the 2^n component simple spinor of $\mathbb{C}\ell_0(2n+2)$, say ϕ'_+ , with the Dirac spinor ϕ of $\mathbb{C}\ell(2n)$ appearing in eq. (4.3) (which is possible because of the isomorphism $\mathbb{C}\ell(2n) \sim \mathbb{C}\ell_0(2n+1) \hookrightarrow 2\mathbb{C}\ell_0(2n+1) = \mathbb{C}\ell(2n+1) \sim \mathbb{C}\ell_0(2n+2)$) we have that the non null vector $Z \in \mathbb{C}^{2n}$ with components $Z_a = z_a^+ + z_a^-$ given by (4.3) is a projection in \mathbb{C}^{2n} of a null vector of \mathbb{C}^{2n+2} whose two extra components are

$$Z_{2n+1} = \langle B\psi, \gamma_{2n+1} \phi \rangle; \quad Z_{2n+2} = \langle B\psi, \pm 1_n \phi \rangle \quad (4.13)$$

provided ϕ may be considered as a simple spinor of $\mathbb{C}\ell_0(2n+2)$, and we have the proposition

Proposition 2. Given two simple spinors φ_{\pm} of $\mathbb{C}\ell_0(2n)$ and the corresponding null vectors $z_{\pm} \in \mathbb{C}^{2n}$ with components

$$z_a^{\pm} = \langle B\psi, \gamma_a \varphi_{\pm} \rangle \quad (4.14)$$

their sum

$$z_a^+ + z_a^- = Z_a = \langle B\psi, \gamma_a \phi \rangle, \quad (4.15)$$

where $\phi = \varphi_+ \oplus \varphi_-$, is a projection on \mathbb{C}^{2n} of a null vector $Z \in \mathbb{C}^{2n+2}$ which is obtained by adding to Z_a the two extra components

$$Z_{2n+1} = \langle B\psi, \gamma_{2n+1} \phi \rangle \quad Z_{2n+2} = \langle B\psi, \pm \phi \rangle \quad (4.16)$$

provided ϕ satisfies the conditions of simplicity as a Weyl spinor of $\mathbb{C}\ell_0(2n+2)$.

In this way we see that to the direct sum of simple or pure spinor spaces, giving rise to a higher dimensional simple spinor space:

$$S_+ \oplus S_- = S, \quad (4.17)$$

where $\mathbb{C}\ell_0(2n) = \text{End}S_{\pm}$ and $\mathbb{C}\ell(2n+2) = \text{End}S$, there correspond the sum of null vectors giving rise to a higher dimensional null vector:

$$z^+ + z^- \hookrightarrow Z \quad (4.18)$$

where $z^{\pm} \in \mathbb{C}^{2n}$ and $Z \in \mathbb{C}^{2n+2}$; and we have an instrument for imbedding spinor spaces in higher dimensional spinor spaces and correspondingly null vector-spaces in higher dimensional null vector spaces.

The above may be easily restricted to the real spaces, of interest for physics, in which case the projective null quadric represent compact manifolds of the type (2.18) or generalizations of them, and then to the direct sums $S_+ \oplus S_- = S$ of simple spinor spaces there will correspond the embedding of compact manifolds into compact ones, increasing at each step the dimension by two. It is interesting to note that for real spaces the signature of the two extra dimensions will be uniquely determined by Proposition 2, since $Z_A Z^A = 0$ is an identity when Z_A are bilinearly expressed in terms of spinors ψ and φ . In particular we adopted it in our constructive approach of spinor spaces, when starting from two component spinors we constructed four component spinors and also the corresponding real, pseudo euclidean vector space, obtained after substituting $B\psi$ with $B\varphi^c = \varphi^{\dagger}$ in eq. (3.4) and obtaining eq. (3.6), with Lorentzian signature unambiguously defined.

Proposition 2 remains valid also for z^\pm and Z real with the only difference that now in eq.(4.16) Z_{2n+1} and Z_{2n+2} have to be substituted with the real $P_{2n+1} = \langle B\phi^c, \gamma_{2n+1}\phi \rangle$ and $P_{2n+2} = \langle B\phi^c, \pm i\phi \rangle$, respectively, and the signature of the $2n + 2$ space is unambiguously defined to remain Lorentzian.

This “construction” is the simplest and the most natural one since strictly correlated with the geometry of simple or pure spinors. Furthermore, it generalizes well established practice with Dirac spinors: each of them is the direct sum of two Weyl two component spinors which are simple spinors of four dimensional space-time. However, the four component Dirac spinor, being direct sum of them, may only be considered simple insofar isomorphic to a Weyl spinor of a six dimensional space. Our “construction” simply consists in extending this natural and well known practice to higher dimensional spaces.

5 FROM FOUR TO EIGHT COMPONENT SPINORS

Let us now sum the two null vectors p_μ^\pm of $\mathbb{R}^{3,1}$ defined by eq. (3.10):

$$p_\mu = p_\mu^+ + p_\mu^- = \tilde{\psi}\gamma_\mu\psi, \quad \mu = 0, 1, 2, 3 \quad (5.1)$$

p_μ , because of Proposition 2, are the projection on $\mathbb{R}^{3,1}$ of a null vector of a six dimensional pseudo-euclidean space, of definite signature, obtained by adding to the 4 real components p_μ the following p_5 and p_6 , both real:

$$p_5 = \tilde{\psi}\gamma_5\psi \quad p_6 = \tilde{\psi}i\psi \quad (5.1')$$

where, as easily verified, p_a ($a = 0, 1, 2, 3, 5, 6$) are real and they identically satisfy the equation

$$p_1^2 + p_2^2 + p_3^2 - p_0^2 + p_5^2 + p_6^2 = 0 \quad (5.2)$$

that is they build up a null vector in $\mathbb{R}^{5,1}$ and their direction define the projective Klein quadric equivalent to S^4 . It may be easily shown that the p_a may be expressed in the following form:

$$p_a = \Psi^\dagger \Gamma_0 \Gamma_a (1 + \Gamma_7) \Psi \quad a = 1, \dots, 6. \quad (5.3)$$

where Γ_a , of the form (4.6), with $\Gamma_6 = \sigma_2 \otimes 1$, are the generators of $\mathbb{C}\ell(5, 1)$ (or of $\mathbb{C}\ell(4, 2)$ if Γ_6 is substituted with $i\Gamma_6$), and Ψ an

associated eight component spinor. With p_a we may generate the Cartan's equation:

$$(p_\mu \gamma^\mu + p_5 \gamma_5 + i p_6) \psi = 0 \quad (5.4)$$

which, for ψ Majorana, since $p_6 \equiv 0$, reduces to:

$$(p_\mu \gamma^\mu + \gamma_5 p_5) \psi = 0 \quad (5.5)$$

where

$$p_\mu p^\mu = -p_5^2 \quad (5.5')$$

and may therefore give origin to the Dirac equation in momentum space $\mathbb{R}^{3,1}$. Obviously there will be another such equation for the projector $(1 - \Gamma_7)$.

Observe that eq. (5.5) may be also directly obtained from the extension of Cartan's eq. (2.1) for Dirac spinors associated with $\mathbb{C}\ell(3, 1)$ to spinors associated with $\mathbb{C}\ell_0(4, 1)$; which is allowed due to isomorphism of simple algebras (2.7'). As such ψ is a Pauli spinor.

We may now also consider the Clifford algebra $\mathbb{C}\ell(4, 2) = \text{End}S$ corresponding to $\mathbb{R}^{4,2}$, the conformal extension of space-time, with generators

$$\Gamma_0, \Gamma_1, \Gamma_2, \Gamma_3, \Gamma_5, \Gamma_6 \quad (5.6)$$

and volume element $\Gamma_7 = -i\Gamma_0\Gamma_1\Gamma_2\Gamma_3\Gamma_5\Gamma_6$. Its simple Weyl spinors π_+, π_- , defined by

$$\pi_\pm = \frac{1}{2} (1 \pm \Gamma_7) \Psi \quad (5.7)$$

where $\Psi \in S$, are vectors of spinor spaces S_\pm such that $\mathbb{C}\ell_0(4, 2) = \text{End}S_\pm$, and obey the Cartan's equation

$$z_a^\pm \Gamma^a (1 \pm \Gamma_7) \Psi = 0 \quad a = 1, 2, \dots, 6 \quad (5.8)$$

where

$$z_a^\pm = \frac{1}{4} \langle B\Psi^c, \Gamma_a \frac{1}{2} (1 \pm \Gamma_7) \Psi \rangle \quad (5.9)$$

and $B\Psi^c = \Psi^\dagger \Gamma_0 \Gamma_6$.

Because of Proposition 1 they are complex ($n = 3$) and precisely

$$z_a^+ = -\bar{z}_a^- \quad (5.10)$$

and therefore eq. (5.8) may not have immediate physical interpretation as real vectors in momentum space; they could instead represent charged currents and as such explain the geometrical origin of the electro-weak model as it will be shown in section 6.2. The Weyl simple spinors π_{\pm} defined by (5.7), associated with $\mathbb{R}^{4,2}$ were named twistors by R. Penrose [9], and have had several interesting applications in mathematics and general relativity.

6 EIGHT COMPONENT SPINORS

6.1 THE NUCLEON DOUBLET

Let us consider the real null vectors given by the generalization of eq. (5.3):

$$p_a^{\pm} = N^{\dagger} \Gamma_0 \Gamma_a (1 \pm \Gamma_7) N \quad a = 1, \dots, 6.$$

where N is an eight component Dirac spinor of $\mathbb{C}\ell(5, 1)$ and let us sum them:

$$\frac{1}{2} (p_a^+ + p_a^-) = p_a = N^{\dagger} \Gamma_0 \Gamma_a N. \quad (6.1)$$

Then, because of Proposition 2, together with the components $p_7 = N^{\dagger} \Gamma_0 \Gamma_7 N$ and $p_8 = N^{\dagger} \Gamma_0 i N$ they build up, for N simple spinor of $\mathbb{C}\ell_0(7, 1)$, a real null vector $P \in \mathbb{R}^{7,1}$ with components:

$$p_A = N^{\dagger} \Gamma_0 \Gamma_A N \quad \text{with} \quad \Gamma_A = \{\Gamma_a, \Gamma_7, -i\} \quad A = 1, \dots, 8. \quad (6.2)$$

In fact if the spinor N is in the Dirac basis (by which we mean that N is the direct sum of two Dirac spinors which is allowed because of the isomorphisms (2.7) and (2.7'))

$$N = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}$$

where ψ_j are Dirac spinors of $\mathbb{C}\ell(3, 1)$, Γ_A have the form:²

$$\Gamma_{\mu} = \begin{pmatrix} \gamma_{\mu} & 0 \\ 0 & \gamma_{\mu} \end{pmatrix}, \quad \Gamma_5 = \begin{pmatrix} 0 & \gamma_5 \\ \gamma_5 & 0 \end{pmatrix}, \quad \Gamma_6 = \begin{pmatrix} 0 & -i\gamma_5 \\ i\gamma_5 & 0 \end{pmatrix},$$

²The spinors ψ_1, ψ_2 in the N doublet may be either Weyl spinors of the automorphism space of $\mathbb{C}\ell_0(5, 1)$, or Pauli of $\mathbb{C}\ell(4, 1)$ or Dirac of $\mathbb{C}\ell(3, 1)$. Correspondingly, either $\Gamma_{\mu} = \sigma_1 \otimes \gamma_{\mu}$ or $\sigma_2 \otimes \gamma_{\mu}$ for Weyl, or $\Gamma_{\mu} = \sigma_3 \otimes \gamma_{\mu}$ for Pauli, or $\Gamma_{\mu} = 1 \otimes \gamma_{\mu}$ for Dirac. These are then $3+1$ independent representations corresponding to the four dimensional space of quaternions.

$$\Gamma_7 = \begin{pmatrix} \gamma_5 & 0 \\ 0 & -\gamma_5 \end{pmatrix} \quad (6.3)$$

and it may be easily verified that, in accordance with Proposition 2, p_A given by (6.2) satisfies to:

$$p_\mu p^\mu + p_5^2 + p_6^2 + p_7^2 + p_8^2 = 0. \quad (6.4)$$

The Cartan's equation for N , taking into account of (6.2) and of (6.3) is:

$$(p_\mu \cdot 1 \otimes \gamma^\mu + \vec{\pi} \cdot \vec{\sigma} \otimes \gamma_5 + iM) N = 0 \quad (6.5)$$

where, according to eq. (6.2):

$$p_\mu = \frac{1}{8}(\tilde{\psi}_1 \gamma_\mu \psi_1 + \tilde{\psi}_2 \gamma_\mu \psi_2); \quad \vec{\pi} = \frac{1}{8} \tilde{N} \vec{\sigma} \otimes \gamma_5 N; \quad M = \frac{i}{8} (\tilde{\psi}_1 \psi_1 + \tilde{\psi}_2 \psi_2) \quad (6.6)$$

where:

$$\tilde{N} = (\tilde{\psi}_1 \quad \tilde{\psi}_2).$$

Eq. (6.5) represents the well-known proton-neutron equation when interacting with the pseudo-scalar pion isovector in momentum space, and with (6.6), eq. (6.5) is an identity. Observe that the pseudoscalar nature of the pion derives from imposition that N is a doublet of Dirac spinors which, in turn, imposes the representation (6.3) of Γ_a where γ_5 must be contained in $\Gamma_5, \Gamma_6, \Gamma_7$ (in order to anticommute with Γ_μ). Again the pion field $\vec{\pi}$ appears here as bilinearly expressed in terms of the proton-neutron field. However this does not imply that in the quantized theory the pion should be considered as a bound state of proton-neutron.

Observe that from eq. (6.6) we have: $p_\mu = \frac{1}{8} (\tilde{\psi}_1 \gamma_\mu \psi_1 + \tilde{\psi}_2 \gamma_\mu \psi_2)$ which could then be interpreted as total momentum of the nucleon doublet (in absence of pions) and as such represent the translation operator giving rise to Poincaré translations and to space-time $\mathbb{R}^{3,1}$, while p_5, p_6, p_7 are extra momenta which represent interaction terms and, in principle, do not need the corresponding dimensions in configuration space. Momentum space is here compact and represented by the projective quadric (6.4); that is S^6/\mathbb{Z}_2 , whose radius, in the physical interpretation may be identified with a mass. We will see in

Chapter 13 that in eq. (6.5) the mass term M has to be set to zero if to N the simplicity or pureness constraint is imposed.

The term $\vec{\pi} \cdot \vec{\sigma} \otimes \gamma_5$ presents the so-called isospin internal symmetry $SU(2)$ of the nucleon doublet which, in view of the above geometrical derivation deserves further comments. We will deal with them in section 7.

6.2 THE ELECTROWEAK MODEL

With eight component spinors we may bilinearly obtain two more real null vectors in eight dimensional spaces. In fact as a consequence of Corollary 1, for $n = 4$, the vectors $K^\pm \in \mathbb{R}^{5,3}$ are real and null. Its real components are [4]:

$$K_A^\pm = \langle \Theta^\dagger G_0 G_6 G_8 G_A (1 \pm G_9) \Theta \rangle, \quad A = 0, 1, 2, 3, 5, 6, 7, 8 \quad (6.7)$$

where G_A are the generators of $\mathbb{C}\ell(5, 3) = \text{End}S$, $\Theta \in S$ and we suppose that the eight component spinors:

$$\Psi_\pm = \frac{1}{2} (1 \pm G_9) \Theta \quad (6.8)$$

are simple Weyl spinors (subject to one constraint equation).

If we take G_A in the Cartan basis, eq. (6.7) may be expressed in the form:

$$K_A^\pm = -\frac{1}{8} \Psi_\pm^\dagger \Gamma_0 \Gamma_6 \Gamma_7 \Gamma_A \Psi_\pm \quad (6.9)$$

where

$$\Gamma_A = \{\Gamma_a, \Gamma_7, \pm 1\},$$

and Γ_a are generators of $\mathbb{C}\ell(4, 2)$.

The corresponding Cartan's equations are:

$$K_A^\pm G_A (1 \pm G_9) \Theta = 0. \quad (6.10)$$

These equations, as well as equation (6.5), might also have physical meaning. To obtain it let us first observe that the components K_A^\pm of the null vector $K^\pm \in \mathbb{R}^{5,3}$ may be also obtained from Proposition 2. In fact let us start from the six dimensional complex null vectors with components z_a^\pm given by (5.9) if we subtract them:

$$z_a^+ - z_a^- = z_a^+ + \bar{z}_a^+ = K_a \quad (6.11)$$

we obtain a real six component vector of $\mathbb{R}^{4,2}$ which identifies with the first six components of K_A given by eq. (6.9), in fact it is easily seen that:

$$K_a = -\frac{1}{8}\Psi^\dagger\Gamma_0\Gamma_6\Gamma_7\Gamma_a\Psi \quad a = 0, 1, 2, 3, 5, 6 \quad (6.12)$$

that is the first six components of K_A^\pm , for $\Psi = \Psi_+$, say: simple spinor of $\mathbb{C}\ell(5, 3)$.

The main difference between the null vectors p_A of the previous section and the K_A of this one is that while p_a are sums of real vectors (see eq. (6.1)) bilinear in Dirac spinor, the K_A are sums of complex vectors (see eq. (6.11)), bilinear in twistors. Therefore their four component space-time vectors may represent neutral and charged currents respectively which may constitute the basic ingredient of the Salam-Weinberg electroweak model, as we will see.

Let us first write the Cartan's equation with K_A :

$$(K_\mu\Gamma^\mu + K_5\Gamma^5 - K_6\Gamma^6 + K_7\Gamma^7 + K_8) \begin{pmatrix} \pi_+ \\ \pi_- \end{pmatrix} = 0 \quad (6.13)$$

where we set $\Psi = \begin{pmatrix} \pi_+ \\ \pi_- \end{pmatrix}$ and by construction π_\pm are Weyl spinors of $\mathbb{C}\ell_0(4, 2)$ or twistors, and therefore the Γ_a are in the Cartan basis (4.6) and we may write eq. (6.13) in matrix form as follows:

$$\begin{pmatrix} K_7 + K_8 & K_\mu\gamma^\mu + K_5\gamma_5 + K_6 \\ K_\mu\gamma^\mu + K_5\gamma_5 - K_6 & -K_7 + K_8 \end{pmatrix} \begin{pmatrix} \pi_+ \\ \pi_- \end{pmatrix} = 0 \quad (6.13')$$

Observe that the first six components of K_A are given by (5.9) where:

$$z_a^\pm = \frac{1}{4}\Psi^\dagger\Gamma_0\Gamma_6\Gamma_a\frac{1}{2}(1 \pm \Gamma_7)\Psi \in M(\pi_\pm)$$

and π_\pm obey the Cartan's equations:

$$(z_\mu^+\gamma^\mu + z_5^+\gamma_5 - z_6^+)\pi_+ = 0 \quad (\bar{z}_\mu^+\gamma^\mu + \bar{z}_5^+\gamma_5 + \bar{z}_6^+)\pi_- = 0. \quad (6.14)$$

Therefore expressing the K_a in (6.13')) as $z_a^+ + \bar{z}_a^+$ and taking into account of (6.14) we arrive at:

$$\begin{pmatrix} \frac{i}{4}\tilde{\pi}_+\pi_+ & \frac{1}{2}(z_\mu\gamma^\mu + z_5\gamma_5 + z_6) \\ \frac{1}{2}(\bar{z}_\mu\gamma^\mu + \bar{z}_5\gamma_5 - \bar{z}_6) & \frac{i}{4}\tilde{\pi}_-\pi_- \end{pmatrix} \begin{pmatrix} \pi_+ \\ \pi_- \end{pmatrix} = 0 \quad (6.13'')$$

where the + superscript for z_a was suppressed and it was taken into account that

$$K_7 = \frac{i}{8} (\tilde{\pi}_+ \pi_+ - \tilde{\pi}_- \pi_-) \quad K_8 = \frac{i}{8} (\tilde{\pi}_+ \pi_+ + \tilde{\pi}_- \pi_-) \quad (6.15)$$

Let us express the complex six-vector z_a in the form

$$z_\mu = A_\mu^{(1)} - iA_\mu^{(2)}, \quad z_5 = \pi_1 - i\pi_2, \quad z_6 = s_2 - is_1 \quad (6.16)$$

where $A_\mu^{(i)}$, π_i , s_i are real. Then taking into account that

$$z_a = \frac{1}{4} \tilde{\pi}_+ \gamma_a \pi_- \quad (6.17)$$

we easily arrive at:

$$\begin{aligned} A_\mu^{(j)} &= \frac{1}{8} \tilde{\Psi} \sigma_j \otimes \gamma_\mu \Psi \\ \pi_j &= \frac{1}{8} \tilde{\Psi} \sigma_j \otimes \gamma_5 \Psi \quad j = 1, 2 \\ s_j &= \frac{1}{8} \tilde{\Psi} \sigma_j \otimes 1 \Psi \end{aligned} \quad (6.18)$$

where $\tilde{\Psi} = (\pi_+^\dagger \gamma_0 \quad \pi_-^\dagger \gamma_0)$.

Setting these in eq. (6.13'') we easily arrive at:

$$\left\{ \sum_{j=1}^2 (A_\mu^j \sigma_j \otimes \gamma^\mu + \pi^j \sigma_j \otimes \gamma_5 + s^j \sigma_j \otimes 1) + M \right\} \begin{pmatrix} \pi_+ \\ \pi_- \end{pmatrix} = 0 \quad (6.13''')$$

where

$$M = \begin{pmatrix} \frac{i}{4} \tilde{\pi}_+ \pi_+ & 0 \\ 0 & \frac{i}{4} \tilde{\pi}_- \pi_- \end{pmatrix} \quad (6.19)$$

We can now go back to the real null vector p_A defined by (6.1) which defines the Cartan's equation (6.5) for the spinor $N = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}$, where now ψ_1, ψ_2 are Dirac spinor of $\mathbb{C}\ell(3, 1)$. Following the same procedure as above we arrive at:

$$[(p_\mu 1 + A_\mu^3 \sigma_3) \otimes \gamma^\mu + \vec{\pi} \cdot \vec{\sigma} \otimes \gamma_5 + M_1] N = 0 \quad (6.20)$$

where

$$p_\mu = \frac{1}{16} (\tilde{\psi}_1 \gamma_\mu \psi_1 + \tilde{\psi}_2 \gamma_\mu \psi_2) \quad A_\mu^3 = \frac{1}{8} \tilde{N} \sigma_3 \otimes \gamma_\mu N$$

and M_1 is given by eq.(6.19) where ψ_1, ψ_2 substitute π_+, π_- .

Now observe that we obtained eq.(6.5), for N in the frame of our programme of constructing fermion multiplets, that is multiplets of $\mathbb{C}\ell(3, 1)$ - Dirac spinors, where we exploited the isomorphism $\mathbb{C}\ell_0(2n+2) \simeq \mathbb{C}\ell(2n+1)$ and consequently obtained $\Psi_D = \psi_D \oplus \psi_D$. With a similar construction we could also have obtained eq.(6.13) for $\Psi = \begin{pmatrix} \pi_+ \\ \pi_- \end{pmatrix}$; we had only to ignore that isomorphism and exploit just the last step of the construction: $\psi_W \oplus \psi_W = \Psi_D$, which means that N and Ψ may be correlated; and in fact define the chiral projectors:

$$L = \frac{1}{2}(1 + \gamma_5); \quad R = \frac{1}{2}(1 - \gamma_5) \quad (6.21)$$

We have that the unitary matrix

$$U = \begin{vmatrix} L & R \\ R & L \end{vmatrix} = U^{-1} \quad (6.22)$$

transforms Ψ in N and vice versa:

$$UN = \begin{vmatrix} L\psi_1 + R\psi_2 \\ R\psi_1 + L\psi_2 \end{vmatrix} = \Psi = \begin{vmatrix} \pi_+ \\ \pi_- \end{vmatrix} \quad (6.23)$$

We will indicate with $\psi_{iL} = L\psi_i$ and with $\psi_{iR} = R\psi_i$ the left-handed and right-handed parts of ψ_i , ($i = 1, 2$) and the same for π_{\pm} . We have then from (6.28):

$$\Psi = \begin{pmatrix} \pi_{+L} \\ \pi_{+R} \\ \pi_{-L} \\ \pi_{-R} \end{pmatrix} = \begin{pmatrix} \psi_{1L} \\ \psi_{2R} \\ \psi_{2L} \\ \psi_{1R} \end{pmatrix}$$

which means (since $LR = 0$) that

$$L\Psi = \Psi_L = LN = N_L \quad (6.24)$$

That is, the left-handed parts of Ψ and of N are identical.

Suppose now that $\psi_1 = e$ and $\psi_2 = \nu_L$ represent an electron and a left-handed massless neutrino respectively. Then let us act with R on eq. (6.13''') and sum it to (6.20) and, remembering that $R\gamma_\mu = \gamma_\mu L$, and that $N = N_L + N_R$ we obtain (after eliminating the pion interaction terms):

$$(p_\mu \gamma^\mu + im) \begin{pmatrix} e \\ \nu \end{pmatrix} + \vec{A}_\mu \cdot \vec{\sigma} \otimes \gamma^\mu \begin{pmatrix} e_L \\ \nu_L \end{pmatrix} + (B_\mu \gamma^\mu + \tau)e_R = 0 \quad (6.25)$$

where $\tau = s_1 - is_2$. \vec{A}_μ is an isotriplet and B_μ a singlet. This equation derives from a Lagrangian which, together with one for the vector fields \vec{A}_μ and B_μ , may be assumed at the basis of the electroweak model.

Observe that the $A_\mu^{(3)}$ component of \vec{A}_μ and B_μ representing neutral vectors is built up from the real vector $p_\mu^{(1)}$ and $p_\mu^{(2)}$ while $A_\mu^{(1)} + iA_\mu^{(2)} = z_\mu$ representing charged vectors responsible of weak interactions are bilinearly constructed with twistors, which might have consequences of interest.

It may be shown [9] that if the triplet $(e_L, e_R, \nu_L) \equiv (e_L, e_L^c, \nu_L)$, where e^c means charge conjugate of e , is subject to an $SU(3)$ symmetry, then the mixing angle θ of the model is fixed such that $\sin^2 \theta = 0.25$.

6.3 THE NEUTRAL-CHARGED FERMION DOUBLETS

The possible geometrical origin of isospin symmetry $SU(2)$ seen above, might also naturally explain the frequent appearance, in the elementary particle landscape, of the charged-neutral fermion doublets like proton-neutron, electron-neutrino, muon-neutrino etc.

Let us in fact write down explicitly eq. (6.5) in terms of the two Dirac spinors ψ_1 and ψ_2 of the N doublet:

$$\begin{aligned} (p_\mu \gamma_\mu + p_7 \gamma_5 + ip_8) \psi_1 + \gamma_5 (p_5 - ip_6) \psi_2 &= 0, \\ (p_\mu \gamma_\mu - p_7 \gamma_5 + ip_8) \psi_2 + \gamma_5 (p_5 + ip_6) \psi_1 &= 0. \end{aligned} \quad (6.5')$$

Now all p_a are real, therefore defining

$$p_5 \pm ip_6 = \rho e^{\pm i \frac{\omega}{2}} \quad (6.26)$$

and multiplying the first eq. (6.5') by $e^{i \frac{\omega}{2}}$ we obtain

$$\begin{aligned} (p_\mu \gamma_\mu + p_7 \gamma_5 + ip_8) e^{i \frac{\omega}{2}} \psi_1 + \gamma_5 \rho \psi_2 &= 0, \\ (p_\mu \gamma_\mu - p_7 \gamma_5 + ip_8) \psi_2 + \gamma_5 \rho e^{i \frac{\omega}{2}} \psi_1 &= 0. \end{aligned} \quad (6.27)$$

We see then that ψ_1 appears with a phase factor $e^{i \frac{\omega}{2}}$ corresponding to a rotation through an angle ω in the circle defined by

$$p_5^2 + p_6^2 = \rho^2 \quad (6.28)$$

in the vector space of the Klein quadric defined by eq. (6.4), which in turn corresponds to an imaginary dilation in $\mathbb{R}^{4,2}$. In fact the corresponding transformation in spinor space is generated by the Lie algebra element

$$J_{56} = \frac{1}{2} [\Gamma_5, \Gamma_6]. \quad (6.29)$$

which is obtained from J_{56} in the $SU(2, 2)$ covering the conformal group, after multiplying the generator Γ_6 by the imaginary unit i .

Observe that this complexification was intrinsic to the modality of our construction which brought us to $\mathbb{C}\ell(5, 1)$ which may be obtained by setting an imaginary unit factor i in front of the generator Γ_6 of $\mathbb{C}\ell(4, 2)$ and which finally generated the $SU(2)$ internal symmetry. In this way $i\Gamma_6$ may be interpreted as a generator of reflections with respect to the sixth (time-like) axis (see Chapter 7) for $\mathbb{C}\ell(4, 2)$.

With this interpretation the dilation covariance of the complexified conformal group induced the $U(1)$ group of symmetry represented by the phase factor in front of ψ_1 and not of ψ_2 (or vice-versa).

If we now translate this in the corresponding Fourier dual Minkowski space-time, since dilation covariance is local, we may consider the phase angle ω as coordinate dependent $\omega \rightarrow \omega(x)$ and then to maintain the covariance of eq. (6.27) we will have to introduce an abelian gauge potential A_μ interacting with ψ_1 only and, as easily seen, the eq. (6.27) will become, in space-time $\mathbb{R}^{3,1}$:

$$\left\{ \gamma_\mu \left[i \frac{\partial}{\partial x_\mu} + \frac{e}{2} (1 - i\Gamma_5\Gamma_6) A_\mu \right] + \vec{\pi} \cdot \vec{\sigma} \otimes \gamma_5 + M \right\} \begin{pmatrix} P \\ N \end{pmatrix} = 0 \quad (6.30)$$

well representing the equation of the proton-neutron doublet interacting with the pion and with the electromagnetic potential A_μ .

Observe that the electric charge Q of the nucleon doublet N is then represented by:

$$Q = \frac{e}{2} (1 - i\Gamma_5\Gamma_6), \quad (6.31)$$

where $-i\Gamma_5\Gamma_6 = \sigma_3 \otimes 1$, that is the third component of isospin generator which, with the other two components $\sigma_1 \otimes 1$ and $\sigma_2 \otimes 1$ generate $SU(2)$ isospin symmetry of nuclear forces.

It is interesting to observe that, in the frame of the study of conformal covariance of spinor field theory, the existence of non equivalent

spinor structure for $\mathbb{C}\ell(4, 2)$ was pointed out by L. Dabrowski [17] and correlated with exotic spinors studied by Petri [18]. Here we see that they may be directly derived from Cartan's equations if interpreted as equations of motion in momentum space and that the local phase factor of the charged partner of the spinor doublet may be interpreted as the result of a complexified dilation.

As we will see this might have interesting consequences also in higher dimensional spaces.

Concluding this chapter we have seen that the minimal Clifford algebras to accommodate eight component spinors are the ones of eight dimensional pseudo-euclidean spaces $\mathbb{R}^{5,3}$ or $\mathbb{R}^{7,1}$ and they give rise to equations (6.5), (6.25) and (6.30) of physical significance in momentum space.

Several more aspects should be further analyzed, among these the simplicity constraint for eight component simple or pure spinors and the role of triality, discovered by Cartan, which poses on equal footing the eight component spinors $\frac{1}{2}(1 \pm G_9)\Theta$ and the null vectors, they will be outlined in Chapter 13. Furthermore it may be expected that octonion field of numbers might play a role, as shown in ref. [10]. In particular in the above derivation of eq. (6.25) the neutral vector A_μ^3 given in eq.(6.20) identifies with equation representing neutral pion π_3 in eq.(6.3) by substituting γ_μ with γ_5 . That is

$$\pi_3 = A_5^3.$$

This could represent the starting point for an attempt of the unification of electroweak and strong interactions in a de Sitter (or anti-de-Sitter) symmetric theory, as will be discussed elsewhere.

7 REFLECTIONS

It is well known that if γ_a are the generators of a Clifford algebra $\mathbb{C}\ell(m, n) = \text{End}S$ corresponding to a pseudo-euclidean space $V = \mathbb{R}^{m, n}$ the commutators $S_{ab} = 1/2 [\gamma_a, \gamma_b]$ are the elements of the Lie algebra of spin (m, n) , the covering group of $SO(m, n)$ acting on S , while the γ_a are operators on S corresponding to reflections with respect to an hyper-plane orthogonal to the unit vector $u_a \in V$. Precisely for any $\psi \in S$ the mentioned reflections induce the transformation [1]:

$$\psi \rightarrow \psi' = \pm \gamma_a \psi. \tag{7.1}$$

Since obviously the square of a reflection must be equal to the identity one must have:

$$\gamma_a^2 = 1. \tag{7.2}$$

Therefore if γ_a represents a time-like unit vector, the corresponding reflection must be represented by $i\gamma_a$.

In Dirac's as well as in Cartan's equations like eq. (2.1) the γ_a appearing in them represent then reflection operators in spinor space.

Let us now consider the space-time conformal group. As well known it may be linearly represented by $SO(4, 2)$ acting on $\mathbb{R}^{4,2}$ with Clifford algebra $\mathbb{C}\ell(4, 2)$ whose generators Γ_a are given by (6.3), where Γ_6 should be substituted by $i\Gamma_6$ (in the Dirac basis), together with the volume element Γ_7 , which extends the Clifford algebra to $\mathbb{C}\ell(5, 2)$. The reflection operators with respect to the corresponding 5th, 6th and 7th hyper-plane will then be represented, in spinor space, by:

$$\Gamma_5, i\Gamma_6, \Gamma_7 \tag{7.3}$$

satisfying the condition (7.2); and these are precisely the operators which appear in eq. (6.5) which give rise to the term $\vec{\pi} \cdot \vec{\sigma} \otimes \gamma_5$ which, formally, is identical to the one traditionally introduced to represent the isospin symmetry $SU(2)$ of pion-nucleon interaction, which then could be ascribed to the reflection operators (7.3), which are reflections of the conformal group (and its volume element). Observe that, when acting in the space of the spinor doublet N , they may be correlated with the quaternion field of numbers (clearly visible for $\mathbb{C}\ell(1, 7)$; see section 11.2, eq. (11.5)).

That the conformal group might be at the origin of isospin symmetry $SU(2)$ is an old idea, based on the fact that in a certain representation (the Dirac one) the eight component spinor associated with $\mathbb{C}\ell(4, 2)$ is represented by a doublet of Dirac spinors, to be identified with the proton-neutron doublet. In fact the idea was first conjectured by W. Heisenberg, the discoverer of isospin. However the obstacle to overcome was twofold. First imbedding $SO(3, 1)$ in $SO(4, 2)$ or, better its covering $SL(2, \mathbb{C})$ in $SU(2, 2)$ one obtains $SU(1, 1)$ and not $SU(2)$. Second the well-known O'Raifeartaigh no-go theorem. In our derivation of the traditional equation (6.5) representing $SU(2)$ isospin symmetry, from spinor geometry, both obstacles are avoided since $SU(2)$ derives from a reflection algebra and because of this the no-go theorem does not apply. Furthermore, as we have seen from spinor geometry,

we naturally derive, through Proposition 2, the real vector p_A given by eq. (6.2) which is null in $\mathbb{R}^{7,1}$ as shown in eq. (6.4).

The fact that the conformal reflections, represented by (7.3), might have an important role, should be welcome to those, like us, who think that the conformal group is a fundamental group of nature, like Maxwell's equation conformal covariance seems to suggest; since otherwise we would have a group in which the first four (space and time) reflections play an important role (parity, antimatter) while the last two (or three) do not.

The validity of conformal symmetry (for massless systems) has induced several authors to conjecture, that Minkowski space-time M may be densely contained in conformally compactified space time $M_c = (S_3 \times S_1) / \mathbb{Z}_2$. There are good arguments [12] to think that, in this case, also the Fourier dual momentum space P should be densely contained in conformally compactified momentum space $P_c = (S_3 \times S_1) / \mathbb{Z}_2$. M_c and P_c build up conformally compactified phase space where both concepts of infinite and infinitesimal (both infrared and ultraviolet divergences) would be *a priori* absent. In this frame conformal reflections could play an important role, of relevance for physics, since they map M_c to P_c and therefore space time to momentum space and vice-versa. And since, as we have shown, momentum space is the appropriate arena for the description of the equations of motion of fermions in first quantization, in spinor geometrical form, while space time is appropriate for the description of classical mechanics in euclidean geometrical form, conformal reflections could help to understand the meaning of the correspondence principle as discussed in ref. [12].

8 SIXTEEN COMPONENT SPINORS

8.1 THE BARYON-LEPTON DOUBLETS

The construction may be continued observing that the real null vector with components p_A given by eq. (6.2) for N , thought as a Weyl spinor of $\mathbb{C}\ell_0(7, 1)$, may be considered as a particular case of of the following equation:

$$P_A^\pm = \Theta^\dagger G_0 G_A (1 \pm G_9) \Theta, \quad A = 1, 2, \dots, 8, \quad (8.1)$$

where Θ is a sixteen component spinor associated with $\mathbb{C}\ell(7, 1)$ of which G_A are the generators and G_9 the volume element.

Again Θ may be considered in the Dirac basis

$$\Theta = \begin{pmatrix} N_1 \\ N_2 \end{pmatrix} \quad (8.2)$$

where N_1 and N_2 are Dirac spinors of $\mathbb{C}\ell(5, 1)$ and, again, defining

$$P_A = P_A^+ + P_A^- = \Theta^\dagger G_0 G_A \Theta \quad A = 1, 2, \dots, 8 \quad (8.3)$$

and

$$\begin{aligned} P_9 &= \Theta^\dagger G_0 G_9 \Theta \\ P_{10} &= \Theta^\dagger i G_0 \Theta \end{aligned} \quad (8.4)$$

we obtain the components P_α of a ten dimensional real null vector which defines the Cartan's equation for the spinor Θ :

$$(P_\mu G^\mu + P_5 G_5 + P_6 G_6 + P_7 G_7 + P_8 G_8 + P_9 G_9 + i P_{10}) \Theta = 0, \quad (8.5)$$

where G_α are the generators of $\mathbb{C}\ell(8, 1)$. For Θ in the Dirac basis we may assign to the G_α the following form:

$$\begin{aligned} G_\alpha &= \left\{ G_a = \begin{pmatrix} \Gamma_a & 0 \\ 0 & \Gamma_a \end{pmatrix}; G_7 = \begin{pmatrix} 0 & \Gamma_7 \\ \Gamma_7 & 0 \end{pmatrix}; \right. \\ G_8 &= \left. \begin{pmatrix} 0 & -i\Gamma_7 \\ i\Gamma_7 & 0 \end{pmatrix}; G_9 = \begin{pmatrix} \Gamma_7 & 0 \\ 0 & -\Gamma_7 \end{pmatrix} \right\} \quad a = 1, 2, \dots, 6. \end{aligned} \quad (8.6)$$

With the same procedure as that of section 6.3, it is easy to show that the Dirac spinors N_1 and N_2 obey a system of equations where N_1 (or N_2) is multiplied by a phase factor $e^{i\tau/2}$, where the angle τ represents a rotation in the circle

$$P_7^2 + P_8^2 = \rho^2 \quad (8.7)$$

for which eq. (8.5) is covariant (in spinor space it is generated by $G_7 G_8$). This $U(1)$ symmetry of N_1 (or N_2) may be interpreted as a charge which, being different from the electric charge (generated by $\Gamma_5 \Gamma_6$), could be the charge of strong forces. In his case then, $N_1 = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}$ could represent the nucleon doublet while $N_2 = \begin{pmatrix} \psi_3 \\ \psi_4 \end{pmatrix}$ the electron neutrino doublet say, both of which contain an electrically charged and neutral component.

We could consider then the strong charge as an eigenvalue of $-iG_7G_8 = \sigma_3 \otimes \mathbb{1}_8$, while the electric charge as an eigenvalue of $-i\Gamma_5\Gamma_6 = \sigma_3 \otimes \mathbb{1}_4$. There is a similarity with Dirac four component spinor ψ conceived as doublet of Weyl φ_+ , φ_- spinors studied in chapter 3. There $\gamma_5 = \sigma_3 \otimes 1_2$ represents chirality, while σ_3 the third component intrinsic angular momentum. As we have seen from eq. (3.11), this component equals $\pm\frac{1}{2}\hbar$ on φ_{\pm} . The corresponding equation for the quadruplet of fermions contained in Θ will be

$$Q_e = \frac{e}{2}(-iG_7G_8 - i\Gamma_5\Gamma_6 \otimes \mathbb{1}) = \begin{pmatrix} +e1_4 & & & \\ & 0 & & \\ & & 0 & \\ & & & -e1_4 \end{pmatrix} \quad (8.8)$$

which could be interpreted as follows: if the charged partner of the baryon doublet N_1 (the proton) has the charge $+e$, the charged partner of the lepton doublet N_2 (the electron) should have the charge $-e$, as it happens, in fact, in nature.

8.2 THE SYMMETRY $SU(2) \otimes U(1)$

If $\Theta = \begin{pmatrix} N_1 \\ N_2 \end{pmatrix}$ is assumed to be in the Pauli representation with N_1 and N_2 Pauli spinors of $\mathbb{C}\ell_0(6,1)$ the generators G_α may be assumed to be:

$$\begin{aligned} G_\mu &= \sigma_3 \otimes 1_2 \otimes \gamma_\mu \\ G_{5,6,7} &= \sigma_2 \otimes \sigma_{1,2,3} \otimes 1_4 \\ G_8 &= \sigma_1 \otimes 1_2 \otimes 1_4 \\ G_9 &= \sigma_3 \otimes 1_2 \otimes \gamma_5 \end{aligned} \quad (8.9)$$

In this case eq. (8.5) takes the form

$$\begin{aligned} (p_\mu 1 \otimes \gamma_\mu + p_9 1 \otimes \gamma_5 + p_{10}) N_1 + (p_8 - i\vec{p} \cdot \vec{\sigma}) N_2 &= 0, \\ (-p_\mu 1 \otimes \gamma_\mu - p_9 1 \otimes \gamma_5 + p_{10}) N_2 + (p_8 + i\vec{p} \cdot \vec{\sigma}) N_1 &= 0, \end{aligned} \quad (8.10)$$

where $\vec{p} \cdot \vec{\sigma} = p_5\sigma_1 + p_6\sigma_2 + p_7\sigma_3$. Then setting

$$(p_8 \pm i\vec{p} \cdot \vec{\sigma}) = \rho e^{\pm\frac{i}{2}\vec{\omega} \cdot \vec{\sigma}} \quad (8.11)$$

where $\rho = \sqrt{p_5^2 + p_6^2 + p_7^2 + p_8^2}$, eq. (8.10) may be written in the form:

$$\begin{aligned} (p_\mu 1 \otimes \gamma_\mu + p_9 1 \otimes \gamma_5 + p_{10}) e^{\frac{i}{2}\vec{\omega} \cdot \vec{\sigma}} N_1 + \rho N_2 &= 0, \\ (-p_\mu 1 \otimes \gamma_\mu - p_9 1 \otimes \gamma_5 + p_{10}) N_2 + \rho e^{\frac{i}{2}\vec{\omega} \cdot \vec{\sigma}} N_1 &= 0, \end{aligned}$$

in which the N_1 doublet manifests an invariance for the phase transformation:

$$N_1 \rightarrow e^{\frac{i}{2}\vec{\omega}\cdot\vec{\sigma}}N_1$$

which appears manifestly as $SU(2)$ covariance for rotations in the unit sphere $u_5^2 + u_6^2 + u_7^2 = 1$, where $\vec{\omega} = \frac{\vec{p}}{|p|}$, with $|p| = \sqrt{p_5^2 + p_6^2 + p_7^2}$. Clearly for local transformation this will give origin to a non abelian Yang-Mills gauge field theory. Θ will then present an $SU(2) \otimes U(1)$ internal symmetry, origin of strong interactions, for the N_1 doublet as seen in eq. (6.5) which will be absent for N_2 , representing the lepton doublet, in which however the geometric structure of the electroweak model will be present as shown in section 6.2.

9 THIRTY TWO COMPONENT SPINORS

Continuing the construction, the P_α of eqs. (8.3) and (8.4) are a particular realization of the following:

$$P_\alpha^\pm = \Phi^\dagger \mathcal{G}_0 \mathcal{G}_\alpha (1 \pm \mathcal{G}_{11}) \Phi \quad \alpha = 1, 2, \dots, 10, \quad (9.1)$$

where Φ is a thirty two component Dirac spinor of $\text{end}\mathbb{C}\ell(9, 1)$ of which \mathcal{G}_α are the generators and \mathcal{G}_{11} the volume element. Then, for Φ simple (as Weyl of $\mathbb{C}\ell_0(11, 1)$)

$$P_\alpha = P_\alpha^+ + P_\alpha^- = \Phi^\dagger \mathcal{G}_0 \mathcal{G}_\alpha \Phi \quad (9.2)$$

together with

$$P_{11} = \Phi^\dagger \mathcal{G}_0 \mathcal{G}_{11} \Phi \quad P_{12} = \Phi^\dagger i \mathcal{G}_0 \Phi \quad (9.3)$$

build up the components of a 12 dimensional real null vector defining the Cartan's equation

$$(P_A \mathcal{G}^A + P_9 \mathcal{G}_9 + P_{10} \mathcal{G}_{10} + P_{11} \mathcal{G}_{11} + iP_{12}) \Phi = 0, \quad (9.4)$$

where $A = 1, 2, \dots, 8$.

For Φ , in the Dirac spinor representation

$$\Phi = \begin{pmatrix} \Theta_1 \\ \Theta_2 \end{pmatrix} \quad (9.5)$$

where Θ_1 and Θ_2 may be considered as Dirac spinors of $\mathbb{C}\ell(7, 1)$, eq. (9.4) will have the form

$$\begin{aligned} (P_\mu G^\mu + P_{11}G_9 + iP_{12}) \Theta_1 + G_9 (P_9 - iP_{10}) \Theta_2 &= 0 \\ (P_\mu G^\mu - P_{11}G_9 + iP_{12}) \Theta_2 + G_9 (P_9 + iP_{10}) \Theta_1 &= 0 \end{aligned} \quad (9.6)$$

and defining

$$(P_9 \pm iP_{10}) = \rho e^{\pm i\frac{\sigma}{2}} \quad (9.7)$$

where $\rho = \sqrt{P_9^2 + P_{10}^2}$, eq. (9.6) may be cast in the form

$$\begin{aligned} (P_\mu G^\mu + P_{11}G_9 + iP_{12}) e^{i\frac{\sigma}{2}} \Theta_1 + \rho G_9 \Theta_2 &= 0 \\ (P_\mu G^\mu - P_{11}G_9 + iP_{12}) \Theta_2 + \rho e^{i\frac{\sigma}{2}} G_9 \Theta_1 &= 0 \end{aligned} \quad (9.6')$$

which presents an $U(1)$ invariance of Θ_1 generated by $\mathcal{G}_9\mathcal{G}_{10}$, corresponding to a rotation through an angle σ in the circle $P_9^2 + P_{10}^2 = \rho^2$. We may interpret it as the $U(1)$ corresponding to a strong charge (or hypercharge). Then the quadruplet of fermions contained in Θ_1 and Θ_2 could represent baryons and leptons respectively.

Observe that each one of them obeys in principle to an equation like (8.5). Furthermore, because of the notorious periodicity theorem of Clifford algebras $\mathbb{C}\ell(k, l)$ we have

$$\mathbb{C}\ell(l + 4, k + 4) = \mathbb{C}\ell(l + 8, k) = \mathbb{C}\ell(l, k + 8) = \mathbb{C}\ell(l, k) \otimes \mathcal{R}(16) \quad (9.8)$$

where $\mathcal{R}(16)$ stands for the algebra of 16×16 real matrices. Therefore, since neither $\mathbb{C}\ell(8, 0)$ nor $\mathbb{C}\ell(0, 8)$ may be associated with real spinors, the final Clifford algebra to study in our construction will be:

$$\mathbb{C}\ell(9, 1) = \mathcal{R}(32) = \mathbb{C}\ell(1, 9), \quad (9.9)$$

admitting real Majorana-Weyl spinors, since after this the geometrical structures, because of the periodicity theorem, will repeat themselves.

Before examining the possibility of representing with Θ_1 and Θ_2 baryons and leptons we wish first to define dimensional reduction procedure in our formulation and to study its meaning.

10 DIMENSIONAL REDUCTION

In our constructive approach, in which at each step we doubled the dimension of our spinor space, while we increased by 2 the dimension of

the correlated vector space, dimensional reduction will simply consist in reversing of those steps.

Precisely, if Ψ is a 2^n component Dirac spinor of the endomorphisms space of a certain Clifford algebra $\mathbb{C}\ell(2n-1, 1)$ with generators γ_a , $a = 1, 2, \dots, 2n$, the dimensional reduction is operated first in spinor space by the chiral projectors $\frac{1}{2}(1 \pm \gamma_{2n+1})$, that is

$$\Psi \rightarrow \varphi_{\pm} = \frac{1}{2}(1 \pm \gamma_{2n+1})\Psi \quad (10.1)$$

where φ_{\pm} are the 2^{n-1} component Weyl spinors of $\mathbb{C}\ell_0(2n-1, 1)$. Correspondingly the $2n+2$ vector space whose vectors with components P_A are constructed bilinearly from Ψ :

$$P_A = \Psi^\dagger \gamma_0 \gamma_A \Psi, \quad A = 1, 2, \dots, 2n+2 \quad (10.2)$$

with $\gamma_{2n+2} = i\mathbb{1}$, will become, after the spinor-space dimensional reduction (10.1):

$$P_A \rightarrow p_a^\pm = \Psi^\dagger \gamma_0 \gamma_a (1 \pm \gamma_{2n+1}) \Psi, \quad a = 1, 2, \dots, 2n, \quad (10.3)$$

They span a $2n$ dimensional vector space since, after the spinor space dimensional reduction, two of the $2n+2$ components P_A and precisely P_{2n+2} and P_{2n+1} , will vanish identically, and dimensional reduction in vector space will be the consequence of dimensional reduction in spinor space. But then, the spinor so reduced, will obey an equation in the lower dimensional momentum-space.

As an example, for $n = 2$, consider the case of four component spinors of Chapter 5. It is easily seen that correspondingly to the chiral projection:

$$\psi \rightarrow \frac{1}{2}(1 \pm \gamma_5)\psi = \varphi_{\pm},$$

which reduces the four component spinor ψ to the two component φ_{\pm} , the null six-vector $p_a = \{p_\mu, p_5, p_6\}$ studied in Chapter 5, reduces to a null four-vector

$$p_\mu^\pm = \tilde{\psi} \gamma_\mu (1 \pm \gamma_5) \psi, \quad \mu = 0, 1, 2, 3,$$

since $p_5^\pm = p_6^\pm = \tilde{\psi} (1 \pm \gamma_5) \psi \equiv 0$.

For $n \geq 3$ the missing terms will generally mean missing of interaction terms in the equations of motion, and the result of “dimensional reduction” simply consists in “decoupling” of the equations of motion.

Obviously depending on the representation of the spinor Ψ and of the γ -matrices, one may have to use other projectors in order to reduce by one half the dimension of spinor space. As an example consider the case of the nucleon doublet dealt with in section 6.1, where $N = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}$ is a doublet of Dirac $\mathbb{C}\ell(3, 1)$ spinors: ψ_1 and ψ_2 . Then, the projectors read $\frac{1}{2}(1 \pm i\Gamma_5\Gamma_6)$ and it is easily seen that these projectors send the 8-dimensional null vector p_A . given by eq. (6.2) into the six dimensional one:

$$p_a^\pm = N^\dagger \Gamma_0 \Gamma_a (1 \pm i\Gamma_5\Gamma_6) N, \quad a = 0, 1, 2, 3, 7, 8.$$

Since $p_{5,6}^\pm = N^\dagger \Gamma_0 \Gamma_{5,6} (1 \pm i\Gamma_5\Gamma_6) N \equiv 0$. Also the equation (6.5) will correspondingly be reduced, missing the terms p_5, p_6 , representing strong interaction terms.

In general, depending from the representation of the generators γ_a of $\mathbb{C}\ell(2n - 1, 1)$ one may build up more projectors in spinor-space each of which will determine a corresponding dimensional reduction in momentum space. Since the γ_a representation determines the representation of the spinor Ψ above, to each such representation there will correspond a dimensional reduction.

In order to determine them, let us first remember that, for the physical interpretation, we need to know the transformation properties both of the spinor-multiplier as well of its components, with respect to the Poincaré group. To this end we need then to determine the form of the first four generators Γ_μ of the concerned Clifford's algebra; that is $\Gamma_1, \Gamma_2, \Gamma_3$ and Γ_0 ; assuming they represent space- and time-directions, respectively.

Now the generators $\Gamma_\mu, (\mu = 1, 2, 3, 0)$ of $\mathbb{C}\ell(2n - 1, 1)$ may have, as known, the following four representations:

$$\begin{aligned} \text{Weyl:} \quad \Gamma_\mu^{(1)} &= \sigma^1 \otimes \gamma_\mu = \begin{pmatrix} 0 & \gamma_\mu \\ \gamma_\mu & 0 \end{pmatrix} \\ \Gamma_\mu^{(2)} &= \sigma^2 \otimes \gamma_\mu = \begin{pmatrix} 0 & -i\gamma_\mu \\ i\gamma_\mu & 0 \end{pmatrix} \\ \text{Pauli:} \quad \Gamma_\mu^{(3)} &= \sigma^3 \otimes \gamma_\mu = \begin{pmatrix} \gamma_\mu & 0 \\ 0 & -\gamma_\mu \end{pmatrix} \\ \text{Dirac:} \quad \Gamma_\mu^{(0)} &= \mathbf{1} \otimes \gamma_\mu = \begin{pmatrix} \gamma_\mu & 0 \\ 0 & \gamma_\mu \end{pmatrix} \end{aligned} \tag{10.4}$$

where γ_μ are the first four generators of $Cl(2n - 3, 1)$.

Corresponding the spinor Ψ associated with $\mathbb{C}\ell(2n-1, 1)$ may be considered as a doublet of Weyl (for $\Gamma_\mu^{(1)}, \gamma_\mu^{(2)}$); Pauli (for $\Gamma_\mu^{(3)}$) or Dirac (for $\Gamma_\mu^{(0)}$) spinors.

For the Clifford algebra $\mathbb{C}\ell(1, 2n-1)$ we would have had instead the generators:

$$\Gamma_\mu^{(j)} = -i\sigma^j \otimes \gamma_\mu; \quad \Gamma_\mu^{(0)} = \mathbf{1} \otimes \gamma_\mu, \quad j = 1, 2, 3 \quad (10.4')$$

Now $-i\sigma^j$ are the known representation of quaternion imaginary units. Therefore, the four representation above may be conceived correlated with quaternion numbers. Now, starting from (10.4) or (10.4') one may easily complete the representation of the generators of $\mathbb{C}\ell(2n-1, 1)$ or $\mathbb{C}\ell(1, 2n-1)$ and we find that the corresponding projects are:

$$\begin{aligned} \text{Weyl:} & \quad \pi_{1,2} = \frac{1}{2} (1 \pm \Gamma_{2n+1}), \quad \text{for } \Gamma_\mu^{(1)} \quad \text{and } \Gamma_\mu^{(2)}, \\ \text{because of which:} & \quad P_{2n+2} \equiv 0 \equiv P_{2n+1}; \\ \text{Pauli:} & \quad \pi_3 = \frac{1}{2} (1 \pm i\Gamma_{2n}\Gamma_{2n+1}), \quad \text{for } \Gamma_\mu^{(3)}, \\ \text{because of which:} & \quad P_{2n+1} \equiv 0 \equiv P_{2n}; \\ \text{Dirac:} & \quad \pi_0 = \frac{1}{2} (1 \pm i\Gamma_{2n-1}\Gamma_{2n}), \quad \text{for } \Gamma_\mu^{(0)}, \\ \text{because of which:} & \quad P_{2n} \equiv 0 \equiv P_{2n-1} \end{aligned} \quad (10.5)$$

The Dirac case may be further extended if one supposes that in $\Gamma_\mu^{(0)}$ of eq. (10.4) also the four γ_μ are in the Dirac representation $\gamma_\mu^{(0)}$; we will indicate it with:

$$\Gamma_\mu^{(00)} = \mathbf{1} \otimes \gamma_\mu^{(0)} = \begin{pmatrix} \gamma_\mu^{(0)} & 0 \\ 0 & \gamma_\mu^{(0)} \end{pmatrix} \quad (10.6)$$

and the corresponding projector will be:

$$\pi_{00} = \frac{1}{2} (1 \pm i\Gamma_{2n-3}\Gamma_{2n-2}), \quad (10.7)$$

because of which

$$P_{2n-2} \equiv 0 \equiv P_{2n-3} \quad (10.8)$$

This may be continued up to $\Gamma_\mu^{(0\dots 0)}$ with m zeros, which we will indicate with $\Gamma_\mu^{(0m)}$ and which will have the corresponding projector:

$$\pi_{0m} = \frac{1}{2} (1 \pm i\Gamma_{2n-2m+1}\Gamma_{2n-2m+2}) \quad (10.9)$$

because of which

$$P_{2n-2m+2} \equiv 0 \equiv P_{2n-2n+1} \quad (10.10)$$

It is easily seen that, given n , the maximal m to be considered is $m = n - 2$ by which the original 2^n Dirac spinor splits in 2^m Dirac spinors each one with 4 components.

When working in space-time in the traditional method, the extra dimensions are introduced in order to explain multiplicities and internal symmetry of fermions in their equations of motions or Lagrangians, and are afterwards eliminated by dimensional reduction, confining them in compact manifold of unobservable size.

Here, instead, the extra dimensions appear as extra terms in the equations of motion in momentum space and, for $n > 2$, they acquire then the meaning of terms representing the interaction of fermions with external fields, as in eq. (6.5). Their elimination may, then, be interpreted as a decoupling, or as the reduction of the equations of the concerned fermions in absence of those interaction terms, a quite natural interpretation since it is natural that a proton (represented by ψ_1 in (6.5) say), being far from a neutron (represented by ψ_2 , say), like those of cosmic rays, when traveling in empty space say, will simply obey the Dirac equation, for a charged fermion.

11 BARYONS AND LEPTONS

11.1 THE BARYON MULTIPLY

Let us now assume in eq. (9.5) the Dirac basis, the 16 component spinor Θ_1 to be of the form:

$$\Theta_1 = \begin{pmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \end{pmatrix} := \Theta_B \quad (11.1)$$

where q_j are Dirac spinors, representing a quadruplets of fermions or quarks, presenting strong charge represented by its $U(1)$ covariance as seen in eq. (9.6'). It obeys eq. (8.5) and P_α given by (8.3), (8.4), being the components of a null vector satisfy identically to (we adopt $\mathbb{C}\ell(1, 9)$):

$$P_\mu P^\mu = P_5^2 + P_6^2 + P_7^2 + P_8^2 + P_9^2 + P_{10}^2 = M^2, \quad (11.2)$$

whose directions define S^5 . Therefore it is to be expected that the quadruplets may present a maximal $SU(4)$ internal symmetry (covering group of $SO(6)$), which will be the obvious candidate for flavor internal symmetry. The most straightforward way to set it in evidence is to determine the 15 generators if the Lie algebra of $SU(4)$ (determined by the commutators $[G_i, G_k]$ for $5 \leq j, k \leq 10$) represented by 4×4 matrices, whose elements are either 1_4 or γ_5 , acting in the space of the Θ_B quadruplet. Let them be λ_j ; where $1 \leq j \leq 15$. Denote with f_j , $1 \leq j \leq 15$ the components of a tensor building up an automorphism space of $SO(6)$. Then, a natural³ equation of motion for Θ_B could be:

$$\left(P_\mu G^\mu + \sum_{j=1}^3 \lambda_j f^j + \sum_{j=4}^8 \lambda_j f^j + \sum_{j=9}^{15} \lambda_j f^j \right) \Theta_B = 0 \quad (11.3)$$

where $\lambda_1 \lambda_2 \lambda_3$ are the generators of $SU(2)$; $\lambda_1, \dots, \lambda_8$ those of $SU(3)$ and $\lambda_1, \dots, \lambda_{15}$ those of $SU(4)$.

In order to study the possible physical information contained in the quadruplet Θ_B , let us now operate with our dimensional reduction.

The most obvious will be to adopt the projector $\frac{1}{2}(1 \pm G_9)$ which is the image in $\mathbb{C}\ell(9, 1)$ of $\frac{1}{2}(1 \pm \mathcal{G}_9 \mathcal{G}_{10})$ of $\mathbb{C}\ell(1, 11)$. Now, the reduction

$$\Theta_B \rightarrow \frac{1}{2}(1 \pm G_9) \Theta_B = N_\pm. \quad (11.4)$$

implies that

$$p_{9,10}^\pm = \Theta_B^\dagger G_0 G_{9,10} (1 \pm G_9) \Theta_B \equiv 0. \quad (11.4')$$

Therefore the null vector P_α given by (8.3) and (8.4) reduces to p_A given by eq. (6.2), which means that N_+ or N_- , conceived as a doublet of fermions obeys eq. (6.5) of the nucleon doublet. It may be shown that if we substitute in that equation the Dirac spinors ψ_1, ψ_2 with Pauli spinors and we operate with the projector π_3 of Chapter 10 in signature (1, 7), the intermediate terms acquires a factor $i1_4$, instead of γ_5 , that is eq. (6.5') becomes a quaternionic equation of the form:

$$(p_\mu \cdot 1 \otimes \gamma^\mu + i\vec{\pi} \cdot \vec{\sigma} + M) N = 0 \quad (11.5)$$

³ In a similar way as $(p_\mu \gamma^\mu + [\gamma_\mu, \gamma_\nu] F^{\mu\nu} + m) \psi = 0$, where $F_{\mu\nu}$ is the electromagnetic tensor, is a natural equation for a fermion, the neutron say, presenting an anomalous magnetic moment.

It is easy to see, that if we operate the dimensional reduction with the projectors $\frac{1}{2}(1 + G_7G_8)$, then $p_{7,8} \equiv 0$, and again the remaining equation is a quaternionic one.

If we now compare these results with the general eq. (11.3) we see that, in all these cases of dimensional reduction, only the internal symmetry represented by the first sum containing $\lambda_1, \lambda_2, \lambda_3$ generators of $SU(2)$ isospin has emerged; which had to be expected in fact, since in the construction of our higher dimensional spinor spaces and of the corresponding vector spaces, where to interpret Cartan's equations as equations of motion in momenta space, the signature resulted unambiguously defined to remain Lorentzian, which in turn determined the emergence of the fundamental role of the quaternion field of numbers. These in turn are at the origin of $SU(2)$ isospin symmetry of the strong nuclear forces, which in fact rules all low energy dynamical phenomena in nuclear physics. This however could not be the end of the story, since, dealing with $\mathbb{C}\ell(1, 9)$ notoriously associated with octonions, presenting the automorphism group G_2 , one could expect to find equations presenting an $SU(3)$ group of symmetry, subgroup to G_2 , represented in eq. (11.3) by the first two sums, as in fact observed in higher energy strong interaction phenomena, when new terms of interaction appear in the equations of motion as will be shown in Chapter 12.

11.2 THE LEPTON MULTIPLET

The multiplet Θ_2 should then not possess strong charge generated by $\mathcal{G}_9\mathcal{G}_{10}$; we will then interpret it as representing leptons:

$$\Theta_2 = \begin{pmatrix} \ell_1 \\ \ell_2 \\ \ell_3 \\ \ell_4 \end{pmatrix} := \Theta_L. \quad (11.6)$$

Since now the strong charge is missing, Θ_L will not obey to eq. (8.5) which implies the possibility of eq.(11.3) and the presence of strong interactions.

The absence of strong charge will impose at least one step of our dimensional reduction: two terms will be missing from eq. (8.5) valid for Θ_B , while the spinor Θ_L will reduce to an 8-component spinor which will be indicated with L . In order to perform this dimensional reduction we will adopt the rules of Chapter 10 for $n = 4$, since Θ_L

may be associated with $\mathbb{C}\ell(1, 7)$. Then we may think their generators to have the form (10.4') where γ_μ are the standard Dirac 4×4 matrices and the projectors are given by (10.5) with $n = 4$. They are:

$$\begin{array}{ll}
\text{Weyl:} & \pi_{1,2} = \frac{1}{2}(1 \pm G_9) \\
\text{because of which:} & P_{10} \equiv 0 \equiv P_9 \\
\text{Pauli:} & \pi_3 = \frac{1}{2}(1 \pm iG_8G_9) \\
\text{because of which:} & P_9 \equiv 0 \equiv P_8 \\
\text{Dirac:} & \pi_0 = \frac{1}{2}(1 \pm G_7G_8) \\
\text{because of which:} & P_8 \equiv 0 \equiv P_7
\end{array}$$

The projectors $\pi_{1,2,3}$ and π_0 will reduce Θ_L to the 8-component spinors $L^{1,2,3}$ and L^0 which will be Weyl-, Pauli- and Dirac-spinors, respectively and the Γ_μ matrices will be the ones of (10.4'). For each of them the momentum space will be 8-dimensional, reduced from the original ten-dimensional one satisfying eq.(11.2) which for $L^{1,2}$ will reduce to:

$$p_\mu p^\mu = p_5^2 + p_6^2 + p_7^2 + p_8^2 = m_{1,2}^2$$

for L^3 will reduce to:

$$p_\mu p^\mu = p_5^2 + p_6^2 + p_7^2 + p_{10}^2 = m_3^2 \quad (11.7)$$

and for L^0 will reduce to:

$$p_\mu p^\mu = p_5^2 + p_6^2 + p_9^2 + p_{10}^2 = m_0^2.$$

Which means, comparing with (11.2), that in principle the masses of the leptons will be lower than those of the baryons, and different among themselves.

Observe further that from what was discussed in section 6.3, in each doublet one lepton should be electrically charged and one neutral.

The lepton multiplet Θ_L should then reduce to 3 doublets L^1, L^2, L^3 , like the N_2 -doublets defined in Section 8.1, labeled by the quaternion imaginary units plus L^0 . If L^1 and L^2 doublets are considered equivalent then we would have only 3 non-equivalent pairs of charged-neutral leptons which might be correlated with the three families of leptons which have been observed in nature.

The one step of dimensional reduction, imposed by the absence of strong charge for the lepton quadruplet Θ_L , will in any case generate the emergence of an $SU(2)$ internal symmetry, which for leptons could be envisaged in the one of the electroweak model discussed in section 6.2.

Up to now we operated with dimensional reduction on the quadruplet Θ_B and Θ_L separately. However it is more natural to operate the reduction directly on the 32-component spinors studied in Chapter 9 and we will obtain the 16-component spinors of Chapter 8.

11.3 THE BARYON-LEPTON QUADRUPLETS

Let us suppose that in the spinor $\Phi = \begin{vmatrix} \Theta_1 \\ \Theta_2 \end{vmatrix}$ of the automorphism space of $\mathbb{C}\ell(1, 11)$ in eq. (9.5), Θ_1 and Θ_2 which we identified as the baryon and lepton multiplets Θ_B and Θ_L respectively, are of the form

$$\Theta_B = \begin{vmatrix} \Theta_{b_1} \\ \Theta_{b_2} \end{vmatrix}, \quad \Theta_L = \begin{vmatrix} \Theta_{\ell_1} \\ \Theta_{\ell_2} \end{vmatrix} \quad (11.8)$$

with Θ_{b_j} and Θ_{ℓ_j} Dirac spinors of $\mathbb{C}\ell(1, 7)$. Then, remembering that dimensional reduction of the lepton quadruplet Θ_L is necessary because of the absence of strong charge, we could reduce the spinors space with the projector

$$\pi_{00} = \frac{1}{2} (1 + \mathcal{G}_7 \mathcal{G}_8) \quad (11.9)$$

by which $P_7 \equiv 0 \equiv P_8$ and the spinor Φ would reduce to

$$\Phi = \begin{vmatrix} \Theta_B \\ \Theta_L \end{vmatrix} \longrightarrow \begin{vmatrix} \Theta_{b_1} \\ 0 \\ \Theta_{\ell_1} \\ 0 \end{vmatrix} = \Theta \quad (11.10)$$

that is to a sixteen component baryon-lepton doublet like those considered in section 8.1:

$$\Theta = \begin{vmatrix} N_1 \\ N_2 \end{vmatrix} = \begin{vmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \\ \psi_4 \end{vmatrix}$$

in which ψ_1, ψ_2 are baryons: we will indicate them with b_1, b_2 and ψ_3, ψ_4 leptons to indicate with ℓ_1, ℓ_2 . We have seen in the preceding section 11.2 that we may obtain them after dimensional reduction from Θ_B and Θ_L in 3 + 1 ways through the projectors $\pi_{1,2,3,0}$ labeled according to the directions of quaternion field of numbers. Therefore the resulting baryon-lepton quadruplet will have the form:

$$\Theta^{(j)} = \begin{pmatrix} b_1^{(j)} \\ b_2^{(j)} \\ \ell_1^{(j)} \\ \ell_2^{(j)} \end{pmatrix}, \quad j = 1, 2, 3, 0 \quad (11.11)$$

The form (11.11) is suggestive since it is apt to set in evidence several of the common properties of baryons and lepton in particular the natural arising of $SU(2) \otimes U(1)$ internal symmetry.

We have to remind that while leptons ℓ_1, ℓ_2 are not subject to strong interactions and therefore for them the dimensional reduction from Θ_L to Θ_{ℓ_1} (or Θ_{ℓ_2}) is necessary, for the baryons b_1, b_2 of the doublet, it is not so since they may be subject to strong interactions. In particular it is known that these interactions present an $SU(3)$ symmetry, and the $SU(3)$ group is notoriously a subgroup of G_2 , the automorphism group of octonions.

We have seen up to now that quaternions seem to play a basic role in the interpretation of some elementary physical phenomena concerning fermions, which may be derived from the geometry of simple or pure spinors; we may then reasonably expect that also octonions or Cayley numbers might play a role, specially because we ended up to deal with spinors associated with the algebras: $\mathbb{C}\ell(1, 1 + 8) = \mathbb{C}\ell(1 + 8, 1)$, notoriously associated with octonions.

12 THE OCTONION FORMALISM

Suppose now the 16 component spinor Θ to be a doublet of $\mathbb{C}\ell_0(5, 1)$ Weyl spinors θ_1, θ_2 : $\Theta = \begin{pmatrix} \theta_1 \\ \theta_2 \end{pmatrix}$. Then G_A , generators of $\mathbb{C}\ell(1, 7)$, may have the form

$$G_a = \begin{pmatrix} 0 & -i\Gamma_a \\ i\Gamma_a & 0 \end{pmatrix}; \quad G_8 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}; \quad G_9 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad a = 1, 2, \dots, 7. \quad (12.1)$$

and eq. (8.5), taking into account of eq. (9.6), reads⁴:

$$P\Theta = \begin{pmatrix} \pm iP_{10} + P_9 & P_8 - i \sum_{j=1}^7 P_j \Gamma_j \\ P_8 + i \sum_{j=1}^7 P_j \Gamma_j & \pm iP_{10} - P_9 \end{pmatrix} \begin{pmatrix} \theta_1 \\ \theta_2 \end{pmatrix} = 0. \quad (12.2)$$

A number of authors [19] have proposed to adopt the formalism of octonion division algebra for the study of physics in ten dimensional momentum spaces of signature (1, 9) and have adopted the equation:

$$P\Theta = \begin{pmatrix} p_{10} + p_9 & p_8 - \sum_{j=1}^7 p_j e_j \\ p_8 + \sum_{j=1}^7 p_j e_j & p_{10} - p_9 \end{pmatrix} \begin{pmatrix} 0_1 \\ 0_2 \end{pmatrix} = 0, \quad (12.3)$$

where e_1, e_2, \dots, e_7 are the anticommutative imaginary units of octonions and $\Theta = \begin{pmatrix} 0_1 \\ 0_2 \end{pmatrix}$ is a two component spinors with 0_1 and 0_2 representing octonions.

Both eqs. (12.2) and (12.3) contemplate the same signature ((9,1) or (1,9)) however while eq. (12.2) implies:

$$P_0^2 - P_1^2 - \dots - P_9^2 - P_{10}^2 = 0, \quad (12.2')$$

eq. (12.3) implies:

$$p_{10}^2 - p_9^2 - \dots - p_1^2 - p_0^2 = 0, \quad (12.3')$$

that is the time components p_0 and p_{10} are interchanged (apart from the \pm sign in front of P_{10} deriving from eq. (9.4) and (9.6)). This means that if in eq. (12.2) we perform the Wick rotations:

$$iP_{10} \rightarrow P_{10}; \quad P_0 \rightarrow iP_0$$

eq. (12.2) and eq. (12.3) must be equivalent. In fact it is known [20] that $\text{Spin}(9, 1) = \text{Spin}(9, 1) = \text{SL}(2, 0) = \text{SL}(32, \mathbb{R})$ where 0 stands for octonions.

Recently T. Dray and C.A. Manogue [19], starting from eq. (12.3) have proposed a dimensional reduction from 10 to 4 dimension in momentum space, adopting the map

$$\pi(q) = \frac{1}{2}(q + \ell q \ell)$$

⁴ Eq. (12.2) may be easily formulated for the signature (1,9); one needs only to substitute $\pm iP_{10}, P_9$ and $i\Gamma_a$ with $\pm P_{10}, iP_9$ and Γ_a respectively, Γ_a being the generators of $\mathbb{C}\ell(1, 5)$.

where q represents an octonion and ℓ a preferred octonion imaginary unit (they use $\ell = e_7$) In this way they obtain from eq. (12.3), spinor equations in 4-dimensional momentum space apt to represent 3 generations of lepton pairs – one massive and one massless in each pair – each generation corresponding to one of the 3 quaternion imaginary units.

Compared to the octonions eq. (12.3) the advantage of eq. (12.2) is that it is straightforward to read the physical meaning of the spinor Θ , especially when we perform dimensional reduction step by step, and furthermore the algebra of the non diagonal terms in eq. (12.2) is the familiar one of Clifford algebras, while that in eq. (12.3) is the non-associative one of octonions. However, in a similar way as the algebra of quaternions seems to play an important role in the explanation of the geometrical structure of some elementary physical phenomena, in particular they might be at the origin of $SU(2) \otimes U(1)$ internal symmetry, and of the 3 families of baryon-lepton quadruplets, the formulation of eq. (12.3) in terms of octonions could be helpful for understanding further aspects of the geometrical structure of some elementary physical phenomena, like the $SU(3)$ (subgroup of G_2) internal symmetry.

To set this in evidence we should then try to correlate eqs. (12.2) and (12.3), that is, to express octonions in terms of Clifford algebra elements. This seems to be indeed possible in the frame of $\mathbb{C}\ell(2, 3)$ and will be extensively analyzed elsewhere [21]. For the moment we will tentatively mention some preliminary results which are coherent with our previous study.

Let us take for the generators of $\mathbb{C}\ell(2, 3)$ the 4×4 matrices

$$\gamma_n = \begin{pmatrix} 0 & -\sigma_n \\ \sigma_n & 0 \end{pmatrix}; \quad \gamma_0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}; \quad \gamma_5 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}; \quad n = 1, 2, 3 \quad (12.4)$$

then we may define the octonion units with

$$e_n = \gamma_n; \quad e_7 = i\gamma_5; \quad e_{n+3} = e_n e_7 = i\gamma_n \gamma_5; \quad n = 1, 2, 3. \quad (12.5)$$

which satisfies the multiplication rules of octonions, that is

$$\begin{aligned} e_\ell e_m &= -\delta_{\ell m} + \varepsilon_{lmn} e_n; & e_\ell e_{m+3} &= -\delta_{\ell m} e_7 - \varepsilon_{lmn} e_{n+3}; \\ e_{\ell+3} e_{m+3} &= -\delta_{\ell m} - \varepsilon_{lmn} e_n; \end{aligned} \quad (12.6)$$

where ε_{lmn} is the emisymmetric tensor, provided the rule for the multiplication of matrices is appropriately modified, as proposed by J. Daboul and R. Delbourgo [22]. It will be shown in ref.[21] that the same result may also be obtained in the frame of the Clifford algebra $\mathbb{C}\ell(2, 3)$ maintaining the standard rule for matrix multiplication.

We will show now that, in the frame of $\mathbb{C}\ell(1, 9)$, there are several sub-algebras apt to represent octonion units as well as their multiplication rules after adoption of the mentioned conventions, and to give origin to equations of motion of the type (11.3) with interaction terms represented by the first two sums.

Let us start from the representation of G_A , generators of $\mathbb{C}\ell(1, 7)$, as given in eqs. (12.1); inserting in them the representations of Γ_a given in eqs. (6.3) one easily finds:

$$G_{4+n} = \begin{pmatrix} 0 & -\sigma_n \\ \sigma_n & 0 \end{pmatrix} \otimes \gamma_5 := e_n \otimes \gamma_5; \quad n = 1, 2, 3 \quad (12.7)$$

where we have adopted the representation (12.4), (12.5) of the octonion units e_n . If we further define

$$iG_9 := e_7 \otimes 1 \quad (12.8)$$

and

$$iG_{4+n}G_9 := ie_n e_7 \otimes \gamma_5 = e_{n+3} \otimes \gamma_5 \quad (12.8')$$

we obtain all the seven imaginary octonion units expressed in the frame of $\mathbb{C}\ell(1, 7)$ which, with the mentioned rules, close the octonion algebra.

Let us now define the complex octonions with:

$$\begin{aligned} v_0 &= \frac{1}{2}(1 - ie_7); & v_n &= \frac{1}{2}(e_n - ie_{n+3}); \\ \bar{v}_0 &= \frac{1}{2}(1 + ie_7); & \bar{v}_n &= \frac{1}{2}(e_n + ie_{n+3}) \end{aligned} \quad (12.9)$$

They satisfy:

$$\begin{aligned} v_0^2 &= v_0; & v_0 \bar{v}_0 &= 0; & v_n v_0 &= \bar{v}_0 v_n = v_n; \\ v_0 v_n &= v_n \bar{v}_0 = 0; & v_l \bar{v}_m &= -\delta_{lm}; & v_l v_m &= \varepsilon_{lmn} \bar{v}_n \end{aligned} \quad (12.10)$$

together with the complex conjugate equations.

This algebra is known to be invariant under the group $SU(3)$ [23]. Precisely (v_1, v_2, v_3) and $(\bar{v}_1, \bar{v}_2, \bar{v}_3)$ transform like $(\mathbf{3})$ and $(\bar{\mathbf{3}})$ representations of $SU(3)$, respectively; while v_0 and \bar{v}_0 like singlets. Therefore it manifests the $SU(3)$ automorphism of the octonion algebra.

It is now easy to express, through eqs. (12.7), (12.8) and (12.8') the complex octonions $v_0, v_n; \bar{v}_0, \bar{v}_n$ in terms of the generators G_{4+n}, G_9 . With them it should then be possible to build up terms transforming as $(\mathbf{3})$ and $(\bar{\mathbf{3}})$ of $SU(3)$ and then presenting on $SU(3)$ internal symmetry. Observe that these terms contain all the projectors $\frac{1}{2}(1 \mp ie_7) = \frac{1}{2}(1 \pm G_9)$. Now if we act with these projectors in spinor space we operate a dimensional reduction, as discussed in section 11.1, and the corresponding equation of motion reduces to the one of the nuclear doublet presenting on $SU(2)$ isospin symmetry.

Let us now extend the above construction to our Clifford algebra $\mathbb{C}\ell(1, 9)$ by defining its generators \mathcal{G}_α as follows:

$$\begin{aligned} \mathcal{G}_\alpha : \mathcal{G}_A &= \begin{pmatrix} 0 & -iG_A \\ iG_A & 0 \end{pmatrix}; \quad \mathcal{G}_9 = \begin{pmatrix} 0 & -G_9 \\ G_9 & 0 \end{pmatrix}, \quad \mathcal{G}_{10} = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \\ \mathcal{G}_{11} &= \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad A = 1, 2, \dots, 8 \quad (12.11) \end{aligned}$$

Then, adopting the above representations for the generators G_A , we easily arrive at the following definitions:

$$\begin{aligned} \mathcal{G}_{6+n} &= \begin{pmatrix} 0 & -\sigma_n \\ \sigma_n & 0 \end{pmatrix} \otimes \Gamma_7 := e_n \otimes \Gamma_7 \quad n = 1, 2, 3 \\ i\mathcal{G}_{11} &:= e_7 \otimes 1 \\ i\mathcal{G}_{6+n}\mathcal{G}_{11} &= ie_n e_7 \otimes \Gamma_7 = e_{n+3}\Gamma_7 \end{aligned} \quad (12.12)$$

which, again, close the octonion algebra with the adoption of the mentioned rules.

In this case the projectors presented by complex octonions will be $\frac{1}{2}(1 \pm \mathcal{G}_{11})$ which, in our model of Chapter 9, splits the 32-component spinor Φ in the baryonic quadruplet Θ_B and the leptonic one Θ_L . Therefore the terms $\mathcal{G}_{6+n}(1 + \mathcal{G}_{11})$ may give rise to $SU(3)$ internal symmetry for Θ_B presenting strong charge while it should be absent for Θ_L (where however it could determine the structure of the electro-weak model determining the value 0.25 of $\sin \theta$, see section 6.2)). This possible $SU(3)$ symmetry for the baryons contains, as seen above, the isospin $SU(2)$ as a subgroup, therefore it would be natural to interpret it as flavour internal symmetry, acting on the baryons interpreted as operators, with all that which follows.

Other possible representations of the octonion units may be found in the frame of $\mathbb{C}\ell(1, 9)$. In fact in Chapter 10 we have seen that in

the frame of $\mathbb{C}\ell(1, 5)$ the first four generators of the algebra may have the form:

$$\Gamma_\mu^{(n)} = \sigma_n \otimes \gamma_\mu \quad \mu = 0, 1, 2, 3 \quad (12.13)$$

with γ_μ generators of $\mathbb{C}\ell(1, 3)$, where $n = 1, 2$ characterize the spinor associated with $\mathbb{C}\ell(1, 5)$ as Weyl doublets, while $n = 3$ as a Pauli doublet. In section 11.3 we have seen a natural reduction of the spinor Φ in 3 families of baryon-lepton quadruplets represented by eq. (11.11). For these the corresponding first four generators of $\mathbb{C}\ell(1, 7)$ could have the form:

$$\Gamma_\mu^{(n)} = \begin{pmatrix} 0 & -\sigma_n \\ \sigma_n & 0 \end{pmatrix} \otimes \gamma_\mu := e_n \otimes \gamma_\mu \quad (12.14)$$

Then with the help of definition (12.8) and of:

$$iG_\mu^{(n)}G_9 := ie_n e_7 \otimes \gamma_\mu = e_{n+3} \otimes \gamma_\mu \quad (12.14')$$

we arrive to another representation of the octonion units, out of which a complex octonions algebra invariant under the $SU(3)$ group may be defined (the same may be repeated with the generators \mathcal{G}_α).

A further representation of octonion units and of the corresponding complex octonion algebra may be found if we take for the generators γ_μ of $\mathbb{C}\ell(1, 3)$ the representation:

$$\gamma_\mu : \gamma_\mu = \begin{pmatrix} 0 & -\sigma_n \\ \sigma_n & 0 \end{pmatrix}, \quad \gamma_0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad n = 1, 2, 3 \quad (12.15)$$

In this case we easily arrive at

$$G_k^{(j)} := e_j \otimes e_k \quad j, k = 1, 2, \dots, 6 \quad (12.16)$$

and

$$-G_7^{(7)} := -G_9 \otimes \gamma_5 := e_7 \otimes e_7 \quad (12.17)$$

which generates, in the complex version, an algebra invariant under the $SU(3) \otimes SU(3)$, which under certain conditions, could be candidate for internal symmetry groups. If they could represent also the color $SU(3)$ as advocated by F. Gürsey [19], is for the moment an open question.

In this way the two first sums appearing in the general eq. (11.3) could be realized. For the possible realization of the last term implying

the, in principle possible, $SU(4)$ internal symmetry in very high energy elementary particle phenomena, there is no sign for the moment. If the parallelism between baryons and leptons, as mentioned in section 11.3, is a dominant feature of elementary particle phenomena then one could expect the discovery of a fourth neutrino, which in fact is being searched for in some laboratories (Fermilab).

Observe that $SU(3)$ is a subgroup of the automorphism group G_2 of octonions and it emerges when among the octonion units a preferred one is chosen; in our case e_7 . This in turn, in our Clifford algebra approach, amounts in choosing a preferred direction in momentum space corresponding to a preferred generator of the concerned Clifford algebra. In the example given above they have been in turn: $G_9, \mathcal{G}_{11}, \gamma_5$. Obviously, also Γ_7 could have been used. These were the volume elements of our even dimensional Clifford algebras and, in the complex octonion algebras, they appear in the form of generalized chiral projectors like $\frac{1}{2}(1 \pm G_9), \frac{1}{2}(1 \pm \mathcal{G}_{11}), \frac{1}{2}(1 \pm \gamma_5)$. This means that in the corresponding $SU(3)$ covariant terms in the Lagrangian densities the corresponding spinors will appear as chirally projected, which might be at the origin of their inobservability.

This possibility is not new in physics where, as an example, the weak current responsible for the neutron decay is $\tilde{P}\gamma_\mu(1 + \gamma_5)N$, and we know that the massive left-handed neutron $(1 + \gamma_5)N$, which appears in the weak interaction term, is not observable as a free particle since it is not invariant under the reflections of the Lorentz group (it obeys to coupled Dirac equations); it behaves as a free particle, only asymptotically in the limit of high momenta, when its mass may be, with good approximation, set to zero.

If the unobservability of colored quarks could have a similar origin, the necessity of conceiving observable fermions as bound states of quarks could be revised (in the above example the neutron does not necessitate to be considered as a bound state of its left- and right-handed Weyl components, which are unobservable as free fermions).

What emerges from this preliminary study is that the geometrical structure of the Clifford algebra $\mathbb{C}\ell(1, 9)$ and of its endomorphism spinor space is very rich and may naturally accommodate several features of the observed elementary natural phenomena of the known fermions.

In particular, all 3 stages of complexity seem to play a fundamental role: complex numbers as origin of the $U(1)$ group of charge in the charged-neutral fermion doublets; quaternions as origin of isospin group of symmetry $SU(2)$ as well as of the 3 families of lepton-baryon

doublets, correlated with the 3 units of quaternions; octonions as the origin of $SU(3)$ internal symmetry perhaps including color, again the number 3 being correlated with the number of quaternion units and also the number 8 of the eightfold representations of $SU(3)$ with the periodicity theorem of Clifford algebras and with the fundamental role of the real algebras $\mathbb{C}\ell(1, 9) = \mathbb{C}\ell(9, 1) = R(32)$ representable through octonions in their automorphism space of Majorana-Weyl spinors.

There are several more geometrical aspects $\mathbb{C}\ell(1, 9)$ and of its endomorphism spinor space which should be further analysed because of their possible correlation with physics. We will try now only to mention some of them.

13 FURTHER GEOMETRICAL ASPECTS

13.1 SIMPLICITY CONSTRAINTS

We based our constructive approach, consisting in embedding spinor spaces and null-vector spaces in higher dimensional ones, on propositions 1 and 2. Both of them deal with simple (or pure) spinors as defined by É. Cartan [1]. Now, for even dimensional Clifford algebras $\mathbb{C}\ell(2n)$, all Weyl spinors are simple for $n \leq 3$; while for $n \geq 4$ they are subject to constraint equations which may be derived from eqs. (2.12).

To start with, for $n = 4$, the 8-dimensional Weyl spinors $\theta_{\pm} = \frac{1}{2}(1 \pm G_9)\Theta$ associated with $\mathbb{C}\ell_0(1, 7)$ are simple iff in eq.(2.12) $F_0 = 0$:

$$F_0 := \langle \theta_{\pm}^t \Gamma_0 \theta_{\pm} \rangle = 0 \quad (13.1)$$

We know that θ_{\pm} may also be conceived as Pauli spinors of $\mathbb{C}\ell(1, 6)$ or as Dirac spinors of $\mathbb{C}\ell(1, 5)$. In this latter case simplicity eq. (13.1) would imply $M = 0$ in eq. (6.5). This condition was sometimes imposed ad hoc in nucleon-pion physics; here we see that it may be obtained imposing the simplicity constraint to the nucleon doublet.

The eight dimensional Weyl spinors $\theta_+ \theta_-$ of $\mathbb{C}\ell(8)$ are specially important since they present a particular outer automorphism, first discovered by Sturdy [23] and named by É. Cartan [1] the triality principle. In fact the 3 eight dimensional quadrics

$$Q_1 = \langle B\theta_+, \theta_+ \rangle, \quad Q_2 = \langle B\theta_-, \theta_- \rangle; \quad Q_3 = X_A X^A, \quad (13.2)$$

where X_A are the orthonormal components of a vector of \mathbb{C}^8 , are obviously invariant with respect to the group $O(8)$ and to its covering group pin (8). The same is obviously true for

$$F = \langle B\theta_+, G_A\theta_- \rangle X^A \quad (13.3)$$

Furthermore, under the transformations:

$$X \leftrightarrow \theta_{\pm}, \quad \theta_+ \leftrightarrow \theta_-, \quad (13.4)$$

F as well as $Q_1 + Q_2 + Q_3$ are invariant. This is the triality automorphism which may be elegantly formulated in the octonion formalism [24]. Eq. (13.1) is a particular case of $Q_1 = 0 = Q_2$, from which, in force of the triality principle, also $Q_3 = P_A P^A = 0$ follows.

The relevance of triality for physics and specially for supersymmetric theories, could be high, however it has not yet been fully clarified.

For $n = 5$, the sixteen component Weyl spinors $\Theta_{\pm} = \frac{1}{2}(1 + \mathcal{G}_{11})\Phi$ associated with $\mathcal{C}\ell_0(1, 9)$ are simple iff in eq. (2.12) $F_1 = 0$, that is iff:

$$F_1 := \langle \Theta_{\pm}^{\dagger}, G_0 G_A \Theta_{\pm} \rangle = 0, \quad A = 1, 2, \dots, 10 \quad (13.5)$$

where $G_{10} := \mathbf{1}$.

These ten constraint equations have been used for an elegant super-Poincaré covariant description and quantization of superstrings [25].

We could also impose simplicity for the 32 component spinors $\Phi_{\pm} = \frac{1}{2}(1 + \mathcal{G}_{13})\Omega$ associated with $\mathcal{C}\ell_0(1, 11)$, for which in eq.(2.12) $F_2 = 0$, that is:

$$F_2 := \langle \Phi_{\pm}^{\dagger}, \mathcal{G}_0 [\mathcal{G}_{\alpha}, \mathcal{G}_{\beta}] \Phi_{\pm} \rangle = 0, \quad \alpha, \beta = 1, 2, \dots, 12 \quad (13.6)$$

where $\mathcal{G}_{12} := \mathbf{1}$.

These 66 constraint equations might also be relevant for physics since we know that Φ_{\pm} may be conceived as Dirac or Pauli spinors associated with $\mathcal{C}\ell(1, 9)$ or with $\mathcal{C}\ell(1, 10)$ respectively, and if we adopt our interpretation of the terms in the equations, with indices $\alpha, \beta \geq 5$, as interaction-terms, then, among the 66 terms set to zero in eq.(13.6), there are several correlating the Θ_B -baryon multiplet with the Θ_L -lepton one. In fact, in the Dirac representation $\mathcal{G}_{\alpha}^{(0)}$ of the generators of $\mathcal{C}\ell(1, 9)$ F_2 implies:

$$\langle \Theta_B^{\dagger} G_0 G_A G_9 \Theta_L \rangle = 0 \quad A = 1, 2, \dots, 8 \quad (13.7)$$

which if G_9 as well as Γ_7 is assumed in the Dirac representation, implies that all weak currents between baryons and leptons are forbidden, and the same may be derived in the Pauli representation. Then the stability of the lightest baryon: the proton might have a geometrical explanation in the frame of simple- or pure-spinor geometry.

13.2 MINIMAL SURFACES AND STRINGS FROM SPINORS

É. Cartan conjectured [1] that simple spinor geometry may underlie euclidean geometry insofar null-vectors may be bilinearly expressed in terms of simple spinors while sums of null-vectors generally give ordinary euclidean vectors.

In the frame of the main trend of theoretical physics of last century to identify more and more with geometry and mathematics, the clue for the implementation of that conjecture might be envisaged in the central role which null-vectors and null-lines (lines with null tangent) played in the last two centuries in the development of geometry in the frame of complex analysis and which emerged in particular from the Enneper-Weierstrass [14] parametrization of minimal surfaces in \mathbb{R}^3 , in the form

$$X_j(u, v) = X_j(0, 0) + \operatorname{Re} \int_c^{u+iv} Z_j(\alpha) d\alpha; \quad j = 1, 2, 3 \quad (13.8)$$

where $X_j(u, v)$ are the orthonormal coordinates of the points of a surface, which is minimal provided $Z_j(\alpha)$ are the holomorphic coordinates of a null \mathbb{C}^3 -vector, and c is any path in the complex plane (u, v) starting from the origin. The correlation with spinors associated with \mathbb{C}^3 are given by eqs.(3.1), (3.2), where $Z_j = \langle B\varphi, \sigma_j \varphi \rangle$ are null. It was shown in ref.[15] that, by considering φ as a Weyl spinor associated with $\mathbb{C}\ell(3, 1)$ eq.(13.8) may easily be extended to $\mathbb{R}^{3,1}$. For a Majorana spinor associated with $\mathbb{C}\ell(3, 1) = R(4)$ the corresponding equation gives the representation of a string in the form:

$$X_\mu(\sigma, \tau) = X_\mu(0, 0) + \int_0^{\sigma+\tau} t_\mu^+(\alpha) d\alpha \int_0^{\sigma-\tau} t_\mu^-(\beta) d\beta; \quad \mu = 0, 1, 2, 3 \quad (13.9)$$

where t_μ^\pm are real, null vectors bilinearly constructed in terms of Majorana spinors. It was further shown [16] that the above formalism may

be extended to higher dimensional Clifford algebras and corresponding spinor spaces and that, whenever Majorana spinors are admitted, strings will be naturally obtained as integrals of bilinear null vectors in terms of them. Propositions 1 and 2 of the present paper may be adopted also in the case of strings such that imbedding simple spinor spaces in higher dimensional ones implies the corresponding imbedding of strings. The above may also be extended to momentum space.

In this way the approach presented in this paper could not only be compatible with, but even be at the origin of string theory; insofar strings could appear as naturally arising bilinearly, from (real) spinors through integrations considered as generalized forms of sums, under certain conditions. It is interesting to note that the condition of reality and simplicity for the corresponding spinors brings to the Clifford algebra

$$\mathbb{C}\ell(9, 1) = \mathbb{C}\ell(8, 0) + \mathbb{C}\ell(1, 1) = R(32), \quad (13.10)$$

and in this frame gravitation will be naturally contained, as well known.

A bilinear parametrization of covariant strings and superstrings theories in terms of Majorana-Weyl spinors associated with $\mathbb{C}\ell(1, 9)$ was recently proposed [24], [25], [26]. The general solutions of the equations of motion are obtained through the use of an octonionic formalism, which might render transparent, through the triality automorphism, also the geometrical origin of supersymmetry.

13.3 FURTHER ASPECTS

There are certainly several more geometrical aspects which need to be further analyzed. Among these the role of triality in supersymmetry theories the possible geometrical origin of coupling constants in the frame of Riemann geometry, the consequent formulation of local field theories in the traditional configuration space and its correlation with momentum space [12] which seems to be the space where spinor geometry may be naturally formulated and immediately interpreted as physical equations of motions; a non surprising fact if one conceives momentum space as the space of velocities.

Also nullness as well as simplicity constraints should be further analyzed since, while introducing the elegant conception of projective manifolds and of compact momentum spaces, they could also furnish sum rules of physical meaning.

14 CONCLUSIONS

Fermion multiplets may be represented by spinor multiplets and, from these, if simple or pure, following É. Cartan, vectors of null quadrics in pseudo euclidean spaces may be bilinearly constructed. It is then natural to try to imbed spinor spaces, and the corresponding bilinearly constructed vector spaces, in higher dimensional ones starting from the most elementary non trivial one: the two-component Pauli spinors of ordinary 3-dimensional space associated with $\mathbb{C}\ell(3)$. It is remarkable that in this construction the signature of the constructed spaces, and of their Clifford algebras, results unambiguously defined starting from $\mathbb{C}\ell(3, 1)$ or $\mathbb{C}\ell(1, 3)$ up to $\mathbb{C}\ell(9, 1)$ or $\mathbb{C}\ell(1, 9)$, after which the sequence is repeated.

It might be not accidental that, on the way of this construction, most of the elementary equations of motion of fermion-physics (including Weyl, and from these, also Maxwell's equations) are naturally found, however in momentum space or the space of velocities, which naturally results compact; and that these equations naturally present some of the main groups of internal symmetry like $U(1)$, $SU(2)$ and $SU(3)$, each one deriving from the 3 degree of complexity of the field of numbers; that is: complex numbers, quaternions and octonions, respectively. This result may be correlated with the general properties of the division algebra like the real $\mathbb{C}\ell(1, 9)$ to which we arrived in our construction as shown by G. Dixon [24].

In this preliminary approach we concentrated our attention on some of the aspects of the geometrical construction and on the possible role of the elegant concept of simple or pure spinors, but much more has to be done specially in the study of its possible relevance for physics, and of the correlation of the resulting compact momentum spaces with figuration space, which seems to be necessary only up to dimension four, the realm of Weyl, Maxwell and Dirac equations and of local field. Beyond four, dimensional reduction appears natural in momentum spaces, where it simply eliminates interaction terms from the equations of motion, while it appears redundant in the corresponding configuration space, for which there seems to be no need, for dimensions larger than four.

References

- [1] *É. Cartan*, “Lecons sur la theorie des spineurs”, Hermann, Paris (1937).
- [2] *C. Chevalley*, “The Algebraic Theory of Spinors”, Columbia U.P., New York (1954).
- [3] *P. Budinich, R. Raczka*, Found. Phys. **23**, 599 (1993).
- [4] *P. Budinich, A. Trautman*, “The Spinorial Chessboard”, Springer, New York (1989).
- [5] *P. Budinich*, Nuovo Cimento **53A**, 31 (1979).
- [6] *P. Budinich*, Found Phys. **23**, 949 (1993).
- [7] *D. Hestenes*, “Space-Time Algebra”, Gordon Breach, New York (1987).
- [8] *J.D. Bjorken*, Ann. of Phys. **24**, 173 (1963);
W.A. Perkins, “Lorentz, CPT and Neutrinos”, World Scientific, Singapore (2000);
K. Just, K. Kwong and Z. Oziewicz, hep-th/005263, 27 May (2000).
- [9] *R. Penrose, M.A.H. Mac Callum*, Phys. Rep. **6c**, 242 (1973).
- [10] *P. Budinich, P. Furlan*, Proceedings of the 2nd Adriatic Meeting, 259, Zagreb (1976);
S. Adler, Phys. Lett. **B225**, 143 (1989).
- [11] *C.A. Manogue, T. Dray*, Mod. Phys. Lett. **A14**, 93 (1999).
- [12] *M. Flato et al.*, Ann. Phys. **61**, 78 (1970);
I.E. Segal, Nuovo Cimento **79B**, 187 (1984).
- [13] *P. Budinich*, Found. Phys. Lett. **12**, 455 (1999).
- [14] *Z. Enneper*, Z. Math. Phys. **9**, 96 (1894);
K. Weierstrass, Monatsberichte der Berliner Akademie **612** (1866).
- [15] *P. Budinich*, Comm. Math. Phys. **107**, 455 (1986).
- [16] *P. Budinich, L. Dabrowski, P. Furlan*, Nuovo Cimento **A96**, 194 (1986);
P. Budinich, M. Rigoli, Nuovo Cimento **B102**, 609 (1988).
- [17] *L. Dabrowski*, Nuovo. Cim. **80A**, 177 (1984).
- [18] *Petri D.R.*, J. Math. Phys. **20**, 231 (1978).

- [19] *F. Gürsey and C.H. Tze*, “On the Role of Division, Jordan, and Related Algebras in Particle Physics”, World Scientific, Singapore (1996);
T. Dray and C.A. Manogue, *Mod. Phys. Lett. A* **14**, 93 (1999).
- [20] *A. Sudbery*, *Journ. Phys.*, **A17**, 939 (1987).
- [21] In preparation.
- [22] *J. Daboul and R. Delbourgo*, *J. Math. Phys.* **40**, 8, 4134 (1999).
- [23] *E. Sturdy*, *Math. Ann.* **39**, 441 (1891).
- [24] *C.A. Manogue and J. Schray*, *J. Math. Phys.* **34**, 3746 (1993);
G.M. Dixon, “Division Algebras: Octonions Quaternions and Complex Numbers and the Algebraic Design of Physics”, Kluwer (1994).
- [25] *N. Berkovitz*, “Cohomology in the Pure Spinor Formalism for the Superstring”, **hep-th 0006003**.
- [26] *D.B. Fairlie and C.A. Manogue*, *Phys. Rev. D.* **26**, 475 (1987).