

FROM THE GEOMETRY OF PURE SPINORS WITH THEIR DIVISION ALGEBRAS TO FERMION'S PHYSICS

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Abstract

The Cartan's equations defining simple spinors (renamed pure by C. Chevalley) are interpreted as equations of motion in momentum spaces, in a constructive approach in which at each step the dimensions of spinor space are doubled while those of momentum space increased by two. The construction is possible only in the frame of the geometry of simple or pure spinors, which imposes constraint equations on spinors with more than four components, and then momentum spaces result compact, isomorphic to invariant-mass-spheres imbedded in each other, since the signatures appear to be unambiguously defined and result steadily lorentzian; starting from dimension four (Minkowski) up to dimension ten with Clifford algebra $\mathbb{C}\ell(1, 9)$, where the construction naturally ends. The equations of motion met in the construction are most of those traditionally postulated ad hoc: from Weyl equations for neutrinos (and Maxwell's) to Majorana ones, to those for the electroweak model and for the nucleons interacting with the pseudoscalar pion, up to those for the 3 baryon-lepton families, steadily progressing from the description of lower energy phenomena to that of higher ones.

The 3 division algebras: complex numbers, quaternions and octonions appear to be strictly correlated with this spinor geometry, from which they appear to gradually emerge in the construction, where they play a basic role for the physical interpretation: at the third step complex numbers generate $U(1)$, possible origin of the electric charge and of the existence of charged – neutral fermion pairs, explaining also the opposite charges of proton-electron. Another $U(1)$ appears to generate the strong charge at the fourth step. Quaternions generate the $SU(2)$ internal symmetry of isospin and the $SU(2)_L$ one, of the electroweak model; they are also at the origin of 3 families; in number equal to that of quaternion units. At the fifth and last step octonions generate the $SU(3)$ internal symmetry of flavour, with $SU(2)$ isospin subgroup and the one of color, correlated with $SU(2)_L$ of the electroweak model. These 3 division algebras seem then to be at the origin of charges, families and of the groups of the Standard model.

In this approach there seems to be no need of higher dimensional (> 4) space-time, here generated merely by Poincaré translations, and dimensional reduction from $\mathbb{C}\ell(1, 9)$ to $\mathbb{C}\ell(1, 3)$ is equivalent to decoupling of the equations of motion.

This spinor-geometrical approach is compatible with that based on strings, since these may be expressed bilinearly (as integrals) in terms of Majorana-Weyl simple or pure spinors which are admitted by $\mathbb{C}\ell(1, 9) = R(32)$.

1 INTRODUCTION

The discovery of fermion - and boson - multiplets which can be labelled according to the representations of certain internal symmetry groups ($SU(2)$; $SU(3)$) brought to the conjecture of the existence of a high dimensional space-time (dim. 10) in which the ordinary one should be imbedded, and the mentioned groups are then interpreted as rotation groups (covering of) in the extra dimensions (> 4). These are then eliminated through “dimensional reduction”; that is by supposing they characterize compact manifolds of very small and unobservable size.

Here we propose a more conservative and economical approach. Reminding that, notoriously, bosons may be bilinearly expressed in terms of fermions, we will start by considering only fermion multiplets, which, as well known, may be represented by spinors.

The geometry of spinors was discovered by É. Cartan [1] who especially stressed the great mathematical elegance of the geometry of those spinors which he named “simple”, subsequently renamed “pure” by C. Chevalley [2]. Our proposal is to adopt it in the study of fermion multiplets. In this spinor-geometrical approach ordinary euclidean vector spaces will first appear in the Cartan’s equations defining spinors, which will be immediately interpretable as equations of motion for the fermions in momentum spaces (or in ordinary Minkowski space-time, provided the first four components of momenta P_μ are interpreted as generators of Poincaré translations). The vectors of such spaces present some attractive properties. First they are null, and their directions define compact manifolds. Furthermore, as shown by Cartan, they may be bilinearly expressed in terms of simple or pure spinors, and, if these represent the fermions, the above equations may become identities in spinor spaces.

These will be the only ingredients for our spinor-geometrical approach which will merely consist in starting from the simplest non trivial two-component Dirac spinors associated with the Clifford algebra $\mathbb{C}\ell(2)$, equivalent to the familiar Pauli $\mathbb{C}\ell_0(3)$ -spinors, and in constructing the fermion multiplets by summing them directly, while building up bilinearly the corresponding vector spaces and equations of motions in momentum-space, step by step, which will be only possible however, in the frame of simple or pure spinor geometry with the use of two Propositions. At each step the dimensions of spinor space will be doubled while those of momentum space will be increased by two. In this construction the signature will result steadily lorentzian while the null momenta will define spheres with radii of the dimension of an energy, or invariant mass. Therefore in the construction the characteristic energies of the phenomena represented will be steadily increasing; as increasing will be the radii of the spheres, and then the characteristic invariant masses.

It is remarkable that in this construction, up to 32 component spinors, (where the construction naturally ends because of the known periodicity theorem on Clifford algebras), one naturally finds, besides Maxwell’s equations, most of those for fermions (otherwise, historically, defined ad hoc) in momentum spaces up to dimension 10, associated with the Clifford algebra $\mathbb{C}\ell(1, 9) = R(32) = \mathbb{C}\ell(9, 1)$, with signatures (including that of Minkowski) and internal symmetry groups, unambiguously well defined in the construction. It is also remarkable that, after the

first step which naturally brings to Weyl equations for massless, two-component, neutrinos, one does not arrive to Dirac's equation for massive fermions, but rather to the equation for massive Majorana fermions represented by four-component real spinors, associated with $\mathbb{C}\ell(3, 1) = \mathbb{R}(4)$. Dirac's equation is here obtained only after the second step, as an approximate equation, when electroweak and/or strong interactions may be ignored.

In this construction each one of the three division algebras: complex numbers, quaternions and octonions, seem to play a fundamental role. In fact the former naturally appears, as soon as a doublet of fermions is reached (at the third step) in the form of a $U(1)$ phase symmetry for one of the two spinors, interpretable as charge of one of the fermions of the doublet, the other being chargeless, reminding electron-neutrino, muon-neutrino, \dots , proton-neutron \dots and interpretable then as electric charge. This phase symmetry, if local, imposes gauge interactions. The next (fourth) step, bringing to two fermion doublets, naturally produces another $U(1)$ for one of the doublets, interpretable then as strong charge, while the quadruplet may well represent most of properties of baryon- and lepton-doublets, including also their similarities and their grouping in 3 families, determined by the 3 imaginary units of the algebra of quaternions. This algebra appears to act on fermion doublets, where it is at the origin of isospin $SU(2)$ symmetry of nuclear forces for the proton-neutron doublet and of the $SU(2)_L$ of the electroweak model, for the lepton doublet. The last and final Clifford algebra $\mathbb{C}\ell(1, 9)$ may be notoriously represented in terms of octonions whose automorphism group G_2 contains a $SU(3)$ subgroup, if one of its seven imaginary units is fixed. We find two such $SU(3)$: one having $SU(2)$ - isospin subgroup, interpretable then as flavour, and an orthogonal one in the dynamical (gauge) sector of the equation interpretable as color, whose subgroup $SU(2)_L$ appears to be correlated with the electroweak model. These symmetries, if local determine gauge interactions (non abelian). Therefore the groups of the standard model: $SU(3)_C \otimes SU(2)_L \otimes U(1)$ seem naturally to originate from these three division algebras.

In this approach the first four components of momentum P_1, P_2, P_3, P_0 may be interpreted as Poincaré generators of space-time, while the other ones $P_5, P_6 \dots P_{10}$ appear, in the equations of motion, as interaction terms (besides the gauge ones of the dynamical sector). Therefore, in this approach there seems to be no need of higher dimension space-time, while dimensional reduction will simply consist in reversing the construction steps, by which first the terms containing P_{10}, P_9 , then P_8, P_7 followed by P_6, P_5 will be eliminated from the equations of motion, which means that dimensional reduction will simply identify with decoupling of the equations of motions, and descending from higher to lower energy phenomena.

2 SIMPLE OR PURE SPINORS

We will briefly summarize here some elements of spinor geometry [1],[2],[3].

Given a $2n$ -dimensional complex space $W = \mathbb{C}^{2n}$ and the corresponding Clifford algebra $\mathbb{C}\ell(2n)$ with generators γ_a obeying $\{\gamma_a, \gamma_b\} = 2\delta_{ab}$, a spinor ϕ is a vector of the 2^n dimensional representation space S of $\mathbb{C}\ell(2n) = \text{End } S$, defined by the

Cartan's equation:

$$z_a \gamma^a \phi = 0; \quad a = 1, 2, \dots, 2n \quad (2.1)$$

where, for $\phi \neq 0, z \in W$ may only be null and defines the Klein quadric Q :

$$Q : \quad z_a z^a = 0 . \quad (2.2)$$

Define $\gamma_{2n+1} := \gamma_1 \gamma_2 \dots \gamma_{2n}$, normalized to one, as the volume element; it anti-commutes with all the γ_a with which it generates $\mathbb{C}\ell(2n+1)$. Weyl spinors ϕ_+ and ϕ_- are defined by

$$z_a \gamma^a (1 \pm \gamma_{2n+1}) \phi_{\pm} = 0 \quad (2.3)$$

they are vectors of the 2^{n-1} -dimensional representation space of the even subalgebra $\mathbb{C}\ell_0(2n)$ of $\mathbb{C}\ell(2n)$.

Given ϕ_{\pm} eq.(2.3) defines a d -dimensional totally null plane $T_d(\phi_{\pm})$ whose vectors are all null and mutually orthogonal. For $d = n$; that is for maximal dimension of the totally null plane $T_n(\phi_{\pm}) := M(\phi_{\pm})$, the corresponding Weyl spinor ϕ_{\pm} is named simple or pure and, as stated by Cartan, $M(\phi_{\pm})$ and $\pm\phi_{\pm}$ are equivalent. All Weyl spinors are simple or pure for $n \leq 3$. For $n \geq 4$ simple or pure Weyl spinors are subject to a number of constraint equations which equals 1, 10, 66, 364 for $n = 4, 5, 6, 7$ respectively (see eq.(2.8)).

To express the vectors $z \in W$ (and tensors) bilinearly in the terms of spinors we need to define the main automorphism B of $\mathbb{C}\ell(2n)$ through:

$$B\gamma_a = \gamma_a^t B; \quad B\phi = \phi^t B \in S^* \quad (2.4)$$

where S^* is the dual spinor space of S , γ_a^t and ϕ^t meaning γ_a and ϕ transposed.

In the case of real, pseudoeuclidean, vector spaces, of interest for physics, we will also need to define the conjugation operator C such that

$$C\gamma_a = \bar{\gamma}_a C \quad \text{and} \quad \phi^c = C\bar{\phi} \quad (2.5)$$

where $\bar{\gamma}_a$ and $\bar{\phi}$ mean γ_a and ϕ complex conjugate.

Another useful definition of simple spinors may be obtained through the formula [4]:

$$\phi \otimes B\psi = \sum_{j=0}^n F_j \quad (2.6)$$

where $\phi, \psi \in S$ are spinors of $\mathbb{C}\ell(2n) = \text{End } S$ and

$$F_j = [\gamma_{a_1} \gamma_{a_2} \dots \gamma_{a_j}] T^{a_1 a_2 \dots a_j} \quad (2.6')$$

where the γ_a products are antisymmetrized and $T^{a_1 a_2 \dots a_j}$ are the components of an antisymmetric j -tensor of \mathbb{C}^{2n} , which can be expressed bilinearly in terms of the spinors ϕ and ψ as follows:

$$T_{a_1 a_2 \dots a_j} = \frac{1}{2^n} \langle B\psi, [\gamma_{a_1} \gamma_{a_2} \dots \gamma_{a_j}] \phi \rangle. \quad (2.7)$$

Setting now $\psi = \phi$, in eq. (2.6) we have that: ϕ is simple or pure if and only if

$$F_0 = 0, \quad F_1 = 0, \quad F_2 = 0, \dots, F_{n-1} = 0 \quad (2.8)$$

while $F_n \neq 0$ and eq. (2.6) becomes:

$$\phi \otimes B\phi = F_n \quad (2.9)$$

and the n -tensor F_n represents the maximal totally null plane $M(\phi)$ of W equivalent, up to a sign, to the simple spinor ϕ . Equations (2.8) represent then the constraint equations for a spinor ϕ associated with W to be a simple or pure. It is easy to verify that for $n \leq 3$, F_0, F_1, F_2 are identically zero for symmetry reasons, while for $n = 4$, the only constraint equation to be imposed is $F_0 = 0$. For $n = 5, 6$ the constraint equations are $F_0 = F_1 = 0$; and $F_0 = F_1 = F_2 = 0$ in numbers 10 and 66 respectively.

The equivalence of this definition with the one deriving from eq. (2.1), given by Cartan, is easily obtained if we multiply eq. (2.6) on the left by γ_a and on the right by $\gamma_a \phi$ and sum over a , obtaining

$$\gamma_a \phi \otimes B\psi \gamma^a \phi = z_a \gamma^a \phi \quad (2.10)$$

where

$$z_a = \frac{1}{2^n} \langle B\psi, \gamma_a \phi \rangle$$

which, provided ϕ is simple or pure, for arbitrary ψ , satisfy

$$z_a \gamma^a \phi = 0$$

and z_a are the components of a null vector of W , belonging to $M(\phi)$, and we have the following proposition:

Proposition 1. Given a complex space $W = \mathbb{C}^{2n}$ with its Clifford algebra $\mathbb{C}\ell(2n)$, with generators γ_a , let ψ and ϕ represent two spinors of the endomorphism spinor-space of $\mathbb{C}\ell(2n)$ and of its even subalgebra $\mathbb{C}\ell_0(2n)$, respectively. Then, the vector $z \in W$, with components:

$$z_a = \langle B\psi, \gamma_a \phi \rangle; \quad a = 1, 2, \dots, 2n \quad (2.11)$$

is null, for arbitrary ψ , if and only if ϕ is a simple or pure spinor.

The proof is given in reference [5].

We have now listed the main geometrical instruments of simple or pure spinor geometry useful in order to proceed with the program.

We will start with the simplest non trivial case of two component spinors, and will perform the first natural step bringing to four component spinors, which, while already well known [6], is here reformulated in order to exhibit transparently some of the main features of Cartan's spinor geometry, and the possible algebraic origin of the Minkowski signature of space-time.

3 THE FIRST NATURAL STEP FROM TWO TO FOUR COMPONENT SPINORS AND THE SIGNATURE OF SPACE-TIME

3.1 Weyl equations

Let us start from $W = \mathbb{C}^2$, with Clifford algebra $\mathbb{C}\ell(2)$, generators σ_1, σ_2 satisfying $\{\sigma_j, \sigma_k\} = 2\delta_{jk}$ and volume element $\sigma_3 = -i\sigma_1\sigma_2$. $\mathbb{C}\ell(2)$ is simple and isomorphic to $\mathbb{C}\ell_0(3) = \text{End } S$ generated by the Pauli matrices $\sigma_1, \sigma_2, \sigma_3$. Its Pauli spinors $\varphi = \begin{pmatrix} \varphi_0 \\ \varphi_1 \end{pmatrix} \in S$ are simple. In fact we have: $B = -i\sigma_2 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} := \epsilon$ and $F_0 = \langle \epsilon\varphi, \varphi \rangle \equiv 0$; eq. (2.9) becomes:

$$\begin{pmatrix} \varphi_0\varphi_1 & -\varphi_0\varphi_0 \\ \varphi_1\varphi_1 & -\varphi_1\varphi_0 \end{pmatrix} \equiv \varphi \otimes B\varphi = z_j\sigma_j \equiv \begin{pmatrix} z_3 & z_1 - iz_2 \\ z_1 + iz_2 & -z_3 \end{pmatrix}. \quad (3.1)$$

Furthermore we have:

$$z_j = \frac{1}{2}\langle B\varphi, \sigma_j\varphi \rangle = \frac{1}{2}\varphi^t \epsilon \sigma_j \varphi \quad (3.1')$$

(compare the elements of the two matrices). From eqs. (3.1) and (3.1'), equation:

$$z_1^2 + z_2^2 + z_3^2 = 0 \quad (3.2)$$

follows and is identically satisfied as may be immediately seen from the determinants of the matrices in eq. (3.1). Also the Cartan's equation:

$$z_j\sigma^j\varphi = 0 \quad (3.3)$$

is identically satisfied, (as may be immediately seen if we act with the first matrix of eq. (3.1) on the spinor $\varphi = \begin{pmatrix} \varphi_0 \\ \varphi_1 \end{pmatrix}$).

If $\psi \in S$ is another spinor we have, from eq. (2.6):

$$\begin{pmatrix} \varphi_0\psi_1 & -\varphi_0\psi_0 \\ \varphi_1\psi_1 & -\varphi_1\psi_0 \end{pmatrix} \equiv \varphi \otimes B\psi = z_0 + z_j\sigma_j \equiv \begin{pmatrix} z_0 + z_3 & z_1 - iz_2 \\ z_1 + iz_2 & z_0 - z_3 \end{pmatrix}, \quad (3.4)$$

and $z_0 = \frac{1}{2}\psi^t \epsilon \varphi$; $z_j = \frac{1}{2}\psi^t \epsilon \sigma_j \varphi$, deriving from it, satisfy identically the equation (as may be immediately seen from the determinants of the matrices):

$$z_1^2 + z_2^2 + z_3^2 - z_0^2 = 0 \quad (3.5)$$

which uniquely determines the signature of Minkowski. In fact the above may be easily restricted to the real by substituting $B\psi$ with $B\phi^c = \phi^\dagger$ by which z_0, z_j become p_0, p_j real:

$$p_0 = \frac{1}{2}\langle \varphi^\dagger \varphi \rangle; \quad p_j = \frac{1}{2}\langle \varphi^\dagger \sigma_j \varphi \rangle \quad (3.6)$$

satisfying identically to:

$$p_1^2 + p_2^2 + p_3^2 - p_0^2 = 0; \quad (3.7)$$

a null or optical vector of Minkowski momentum space $P = \mathbb{R}^{3,1}$, which then originates from the structure of Pauli matrices (or quaternion units): see the second matrix in (3.4) or, equivalently, from the Clifford algebras isomorphism:

$$\mathbb{C}\ell(3) \simeq \mathbb{C}\ell_0(3, 1), \quad (3.8)$$

after which φ may be interpreted as a simple Weyl spinor associated with $\mathbb{C}\ell_0(3, 1) = \text{End } S_{\pm}$; and, since $\mathbb{C}\ell_0(3, 1)$ is not simple, there are two of them: φ_+, φ_- satisfying the Cartan's equations:

$$(\vec{p} \cdot \vec{\sigma} + p_0) \varphi_+ = 0 \quad (\vec{p} \cdot \vec{\sigma} - p_0) \varphi_- = 0. \quad (3.9)$$

These equations may be expressed as a single equation for the four component Dirac spinor $\psi = \varphi_+ \oplus \varphi_-$. In fact indicating with

$$\gamma_\mu = \{\gamma_0; \gamma_j\} := \{-i\sigma_2 \otimes 1; \sigma_1 \otimes \sigma_j\}; \quad j = 1, 2, 3$$

the generators of $\mathbb{C}\ell(3, 1)$ and with $\gamma_5 = -i\gamma_0\gamma_1\gamma_2\gamma_3 = \sigma_3 \otimes 1$ its volume element, we may write (3.9) in the form

$$p_\mu^\pm \gamma^\mu (1 \pm \gamma_5) \psi = 0 \quad (3.9')$$

where the Weyl spinors $\varphi_\pm = \frac{1}{2}(1 \pm \gamma_5) \psi$ are simple or pure; they are eigenspinors of γ_5 .

The optical vectors p_μ^\pm may be expressed in the form:

$$p_\mu^\pm = \frac{1}{2} \tilde{\psi} \gamma_\mu (1 \pm \gamma_5) \psi, \quad (3.10)$$

where $\tilde{\psi} = \psi^\dagger \gamma_0$; with which eq.(3.7), becomes an identity. Eq.(3.9') is a particular case of Cartan's eq.(2.3) defining simple or pure spinor and as such it may be dealt with as an algebraic equation¹. To interpret it as an equation of motion we have to postulate that the spinor $\psi := \psi(p)$ is a function of $p \in P = \mathbb{R}^{3,1}$ (one could conceive the Clifford algebra $\mathbb{C}\ell_0(3, 1)$ as a fiber in $P = \mathbb{R}^{3,1}$) and then eqs.(3.9') represent the Weyl equations for massless neutrinos in momentum space P .

But one could also think of p_μ as generators of Poincaré translations through which we may notoriously generate, as a homogeneous space, Minkowski space-time $M = \mathbb{R}^{3,1}$. Then if $x \in M$, $p_\mu = i \frac{\partial}{\partial x_\mu}$ and $\psi := \psi(x)$ (again we could conceive $\mathbb{C}\ell_0(3, 1)$ as a fiber in M) and eqs.(3.9') become

$$i \frac{\partial}{\partial x_\mu} \gamma^\mu (1 \pm \gamma_5) \psi(x) = 0 \quad (3.11)$$

that is Weyl equations for massless neutrinos in space-time M . These two options are both mathematically correct. In this case they are equivalent since eq.(3.9')

¹One may define spinors as minimal left ideals of Clifford algebras and then, in the Witt nilpotent basis of $\mathbb{C}\ell(3, 1)$, simple or pure spinors are products of basis elements as $\psi_0 = n_1 n_2$ say, with $n_1 = \frac{1}{2}(\gamma_1 + i\gamma_2)$; $n_2 = \frac{1}{2}(\gamma_3 + \gamma_0)$; and eq.(3.9') becomes $(x_1 n_1 + x_2 n_2) \psi_0 = 0$ satisfied for any $x_1 = p_1 - ip_2$ and $x_2 = p_3 - p_0$ [4].

is the Fourier transform of eq.(3.11). Observe that in this approach space-time appears merely as generated by Poincaré translations.

If p_μ^\pm given in eq.(3.10) is inserted in the equation of motion (3.9'), this appears expressed in terms of simple or pure spinors associated with $\mathbb{C}\ell_0(3,1)$ (and it can also become an identity) and therefore establishes a correlation of that geometry with the physics of massless neutrinos (like non conservation of parity). The solution $\psi(x)$ of eq.(3.11) describes the space-time behaviour of a massless neutrino in a particular phenomenon: the fact that, if inserted in (3.11), it makes it an identity means that such description is valid for any space and time coordinate. These properties of Cartan's eqs.(3.9') and (3.11) are general and valid also for the other equations we will deal with in the paper.

3.2 Maxwell's equations

Also Maxwell's equations may be easily obtained from eq.(3.9'). In fact the Weyl spinors φ_\pm are simple and therefore with them eq. (2.9) becomes:

$$\varphi_\pm \otimes B\varphi_\pm = \frac{1}{2}F_\pm^{\mu\nu} [\gamma_\mu, \gamma_\nu] (1 \pm \gamma_5) \quad (3.12)$$

where the antisymmetric tensor $F_\pm^{\mu\nu}$ may be expressed bilinearly in terms of spinors through eq. (2.7):

$$F_\pm^{\mu\nu} = \tilde{\psi} [\gamma^\mu, \gamma^\nu] (1 \pm \gamma_5) \psi, \quad (3.13)$$

and $F_\pm^{\mu\nu}$, because of eq.(3.9') satisfy the homogeneous Maxwell's equations [7],[8]:

$$p_\mu F_+^{\mu\nu} = 0; \quad \epsilon_{\lambda\rho\mu\nu} p^\rho F_-^{\mu\nu} = 0. \quad (3.14)$$

Observe that from eq.(3.13) it appears that the electromagnetic tensor $F_{\mu\nu}$ is bilinearly expressed in terms of the Weyl spinors φ_+ and φ_- obeying the equations of motion (3.9') of massless neutrinos. This however does not imply that, in the quantized theory, the photon must be conceived as a bound state of neutrinos. In fact it is known that the neutrino theory of light, while violating both gauge invariance and statistics, is unacceptable [9].

4 IMBEDDING SPINORS AND NULL-VECTOR SPACES IN HIGHER DIMENSIONAL ONES

Our programme is to imbed spinor spaces in higher dimensional ones, and this is easy with the instrument of direct sums; but we want also to imbed the corresponding null vector spaces in higher dimensional ones, which might appear difficult since our momentum space vectors are bilinearly constructed from spinors and furthermore, sums of null vectors are not null, in general. This will be possible instead only in the frame of simple- or pure-spinors geometry, as we will see.

Let us start by considering the isomorphisms of Clifford algebras:

$$\mathbb{C}\ell(2n) \simeq \mathbb{C}\ell_0(2n+1) \quad (4.1)$$

both being central simple [3], and

$$\mathbb{C}\ell(2n+1) \simeq \mathbb{C}\ell_0(2n+2) \quad (4.2)$$

both non simple. From these we may have the following imbeddings:

$$\mathbb{C}\ell(2n) \simeq \mathbb{C}\ell_0(2n+1) \hookrightarrow \mathbb{C}\ell(2n+1) \simeq \mathbb{C}\ell_0(2n+2) \hookrightarrow \mathbb{C}\ell(2n+2). \quad (4.3)$$

We will indicate with ψ_D, ψ_P , and ψ_W the spinors associated with $\mathbb{C}\ell(2n)$, $\mathbb{C}\ell_0(2n+1)$ and $\mathbb{C}\ell_0(2n)$ respectively, where D, P and W stand for Dirac, Pauli and Weyl. Then to the embeddings (4.3) of Clifford algebras there correspond the following spinor embeddings:

$$\psi_D \simeq \psi_P \hookrightarrow \psi_P \oplus \psi_P \simeq \psi_W \oplus \psi_W = \Psi_D \simeq \psi_D \oplus \psi_D \quad (4.4)$$

which means that 2^n component Dirac spinor is isomorphic to a 2^n component Pauli spinor and that the direct sum of two such Pauli spinors, equivalent to that of two Weyl spinors may give a Dirac spinor with 2^{n+1} components, which may be then considered as a doublet of 2^n Dirac spinors, which is in accordance with our programme.

These isomorphisms may be formally represented through unitary transformations in spinor spaces. In fact indicating with Γ_A and γ_a the generators of $\mathbb{C}\ell(2n+2)$ and of $\mathbb{C}\ell(2n)$ respectively (we adopt the symbols of Appendix A1 corresponding to $n=2$), assume for the first $2n$ generators of Γ_A the following forms:

$$\Gamma_a^{(0)} = 1_2 \otimes \gamma_a; \quad \Gamma_a^{(j)} = \sigma_j \otimes \gamma_a; \quad j = 1, 2, 3; \quad a = 1, 2 \dots 2n \quad (4.5)$$

where $\sigma_1, \sigma_2, \sigma_3$ are Pauli matrices; then if

$$\Psi = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} \quad (4.6)$$

is a 2^{n+1} -component spinor associated with $\mathbb{C}\ell(2n+2)$, the 2^n -component spinors ψ_1 and ψ_2 are:

$$\begin{aligned} \mathbb{C}\ell(2n) - \text{Dirac spinors for } \Gamma_a^{(0)} & \quad \text{and we indicate } \Psi := \Psi^{(0)} \\ \mathbb{C}\ell_0(2n+2) - \text{Weyl spinors for } \Gamma_a^{(1)} \text{ or } \Gamma_a^{(2)} & \quad \text{and we indicate } \Psi := \Psi^{(1)} \text{ or } \Psi^{(2)} \\ \mathbb{C}\ell(2n+1) - \text{Pauli spinors for } \Gamma_a^{(3)} & \quad \text{and we indicate } \Psi := \Psi^{(3)}. \end{aligned} \quad (4.6')$$

Given $\Gamma_a^{(m)}$ with $m = 0, 1, 2, 3$, as in eq.(4.5), the other two generators and the volume element are:

$$\begin{aligned} \Gamma_{2n+1}^{(0)} &= \sigma_1 \otimes \gamma_{2n+1}; & \Gamma_{2n+2}^{(0)} &= \sigma_2 \otimes \gamma_{2n+1}; & \Gamma_{2n+3}^{(0)} &= \sigma_3 \otimes \gamma_{2n+1} \\ \Gamma_{2n+1}^{(1)} &= \sigma_1 \otimes \gamma_{2n+1}; & \Gamma_{2n+2}^{(1)} &= \sigma_2 \otimes 1; & \Gamma_{2n+3}^{(1)} &= \sigma_3 \otimes 1 \\ \Gamma_{2n+1}^{(2)} &= \sigma_2 \otimes \gamma_{2n+1}; & \Gamma_{2n+2}^{(2)} &= \sigma_1 \otimes 1; & \Gamma_{2n+3}^{(2)} &= \sigma_3 \otimes 1 \\ \Gamma_{2n+1}^{(3)} &= \sigma_3 \otimes \gamma_{2n+1}; & \Gamma_{2n+2}^{(3)} &= \sigma_2 \otimes 1; & \Gamma_{2n+3}^{(3)} &= \sigma_1 \otimes 1, \end{aligned} \quad (4.7)$$

where all the diagonal unit matrices indicated with 1 have dimension 2^n .

Define now the projectors:

$$L := \frac{1}{2}(1 + \gamma_{2n+1}); \quad R := \frac{1}{2}(1 - \gamma_{2n+1}) \quad (4.8)$$

and the unitary transformations:

$$U_j := 1_2 \otimes L + \sigma_j \otimes R = U_j^{-1}; \quad j = 1, 2, 3. \quad (4.9)$$

Then it is easily seen that

$$U_j \Gamma_A^{(0)} U_j^{(-1)} = \Gamma_A^{(j)}; \quad A = 1, 2, \dots, 2n+2; \quad j = 1, 2, 3 \quad (4.10)$$

and therefore

$$U_j \Psi^{(0)} = \Psi^{(j)}; \quad j = 1, 2, 3 \quad (4.11)$$

which formally proves the isomorphisms of eq.(4.4). The above may be easily extended to pseudo-euclidean, in particular lorentzian, signatures.

Observe that the doublets $\Psi^{(m)}$ are isomorphic according to eqs.(4.4) and (4.11), however physically inequivalent, since Weyl spinors (or twistors [6]) are not invariant for space-time reflections and therefore ψ_1^W, ψ_2^W may not be observed as free fermions (if massive), but may only enter in interactions (weak ones, say). In fact, as we will see in section 6.3, the U_j of eq.(4.9) will give rise to $SU(2)_L$ of the electroweak model.

In order to perform the proposed doubling of dimensions of simple or pure spinor space while increasing also by two the dimension of the corresponding null-vector space, we need the following Proposition:

Proposition 2. Let ϕ_D and ψ_D represent two spinors associated with $\mathbb{C}\ell(2n)$, with generators γ_a , and let the Weyl spinors $\frac{1}{2}(1 \pm \gamma_{2n+1})\psi_D$ be simple or pure, then, because of Proposition 1, the vectors $z^\pm \in \mathbb{C}^{2n}$ with components

$$z_a^\pm = \frac{1}{2} \langle B\phi_D, \gamma_a(1 \pm \gamma_{2n+1})\psi_D \rangle \quad (4.12)$$

are null and their non null sum $z = z^+ + z^-$ with components

$$z_a = z_a^+ + z_a^- = \langle B\phi_D, \gamma_a\psi_D \rangle \quad (4.13)$$

is the projection from \mathbb{C}^{2n+2} to \mathbb{C}^{2n} of the vector $Z \in \mathbb{C}^{2n+2}$ with components

$$Z_a = z_a; \quad Z_{2n+1} = \langle B\phi_D, \gamma_{2n+1}\psi_D \rangle; \quad Z_{2n+2} = \langle B\phi_D, \psi_D \rangle \quad (4.14)$$

which is null, provided ψ_D , conceived as a Weyl spinor of $\mathbb{C}\ell_0(2n+2)$, is simple or pure.

The proof is given in ref.[8].

The above may be restricted to real spaces of lorentzian signature:

Corollary 2. Let ψ represent a spinor associated with $\mathbb{C}\ell(2n-1, 1)$, with generators γ_a , and let its Weyl spinors $\frac{1}{2}(1 \pm \gamma_{n+1})\psi$ be simple or pure, then the vectors $p^\pm \in \mathbb{R}^{2n-1,1}$, with components:

$$p_a^\pm = \frac{1}{2}\tilde{\psi}\gamma_a(1 \pm \gamma_{2n+1})\psi \quad (4.15)$$

where $\tilde{\psi} = \psi^\dagger \gamma_0$, are null and their non null sum $p = p^+ + p^-$ is the projection from $\mathbb{R}^{2n+1,1}$ in $\mathbb{R}^{2n-1,1}$ of a vector $P \in \mathbb{R}^{2n+1,1}$ with real components:

$$P_a = p_a; \quad P_{2n+1} = \tilde{\psi}\gamma_{2n+1}\psi; \quad P_{2n+2} = i\tilde{\psi}\psi \quad (4.16)$$

which is null, provided ψ , conceived as a Weyl spinor of $\mathbb{C}\ell_0(2n+1, 1)$, is simple or pure.

After this we know that, after the first step of Chapter 3 which gave us Minkowski signature, our construction will give us only lorentzian signatures.

Observe that the imaginary unit i in front of $\tilde{\psi}\psi$ is necessary in order to render P_{2n+2} real for ψ complex (for ψ real P_{2n+2} is identically null, as may be easily verified). For the Clifford algebra $\mathbb{C}\ell(1, 2n-1)$ Corollary 2 is also true but in eqs.(4.16) we have now to insert $P_{2n+1} = -i\tilde{\psi}\gamma_{2n+1}\psi$ and $P_{2n+2} = \tilde{\psi}\psi$, both real (see Appendix A1.).

The vector $P \in \mathbb{R}^{2n+1,1}$ with real components $P_A = \{p_a; P_{2n+1}, P_{2n+2}\}$ will give rise to a Cartan's equation for the spinor Θ associated with $\mathbb{C}\ell(2n+1, 1)$ with generators G_A (see Appendix A1 from which we adopt the symbols for $n=3$):

$$P^A G_A (1 \pm G_{2n+3})\Theta = 0, \quad (4.17)$$

which if

$$\Theta = \begin{pmatrix} \Psi_+ \\ \Psi_- \end{pmatrix} \quad (4.18)$$

gives the four equivalent equations:

$$\left(p^a \Gamma_a^{(m)} + P_{2n+1} \Gamma_{2n+1}^{(m)} \pm P_{2n+2} \right) \Psi_\pm^{(m)} = 0; \quad m = 1, 2, 3, 0, \quad (4.19)$$

where $\Gamma^{(m)}$ and $\Psi^{(m)}$ are defined by eqs.(4.5) and (4.6') and $\Psi^{(m)}$ are isomorphic as shown by eq.(4.11).

We have now the geometrical instruments appropriate to proceed in our construction consisting in doubling the dimensions of simple or pure spinor spaces: $S_+ \oplus S_- = S$ and of increasing by two the dimension of the corresponding null vector spaces: $z_+ \oplus z_- \hookrightarrow Z$ or $p_+ \oplus p_- \hookrightarrow P$ at each step.

5 THE SECOND STEP

Majorana equation

Let us now sum the two null vectors p_μ^\pm of $\mathbb{R}^{3,1}$ defined by eq. (3.10):

$$p_\mu = p_\mu^+ + p_\mu^- = \tilde{\psi}\gamma_\mu\psi, \quad \mu = 0, 1, 2, 3 \quad (5.1)$$

because of Corollary 2, they are the projection on $\mathbb{R}^{3,1}$ of a null vector of a six dimensional space, obtained by adding to the 4 real components p_μ the following p_5 and p_6 , both real:

$$p_5 = \tilde{\psi}\gamma_5\psi \quad p_6 = i\tilde{\psi}\psi \quad (5.1')$$

where, as easily verified, p_a identically satisfy the equation

$$p_1^2 + p_2^2 + p_3^2 - p_0^2 + p_5^2 + p_6^2 = 0 \quad (5.2)$$

that is they build up a null vector in $\mathbb{R}^{5,1}$ and their directions define the projective Klein quadric equivalent to S^4 .

We see that, in accordance with Corollary 2, the signature of the momentum vector space remains lorentzian. Now since $\mathcal{Cl}(3, 1) = \mathbb{R}(4)$ we may represent it in the space of real spinors ψ . But then $p_6 \equiv 0$ and Cartan's equation becomes:

$$(p_\mu\gamma^\mu + \gamma_5 m)\psi = 0 \quad (5.3)$$

that is Majorana equation. For ψ complex instead the equation would be:

$$(p_\mu\gamma^\mu + \gamma_5 p_5 + ip_6)\psi = 0. \quad (5.4)$$

6 THIRD STEP: ISOTOPIC SPIN $SU(2)$, THE ELECTRIC CHARGE AND THE ELECTRO-WEAK MODEL

6.1 Pion-nucleon equation

Let Ψ represent an 8-component Dirac spinor associated with $\mathcal{Cl}(5, 1)$ generated by Γ_a , the vectors $p^\pm \in \mathbb{R}^{5,1}$ with components

$$p_a^\pm = \frac{1}{2}\Psi^\dagger\Gamma_0\Gamma_a(1 \pm \Gamma_7)\Psi, \quad a = 1, 2, \dots, 6 \quad (6.1)$$

are null because of Proposition 1 (since $(1 \pm \Gamma_7)\Psi$ are simple). Eqs.(5.1) and (5.1') are a particular case of these (see A1, for $\mathcal{Cl}(1, 5)$ and $\Gamma_a^{(1)}$).

Consider the components: $p_a^+ + p_a^- = p_a = \Psi^\dagger\Gamma_0\Gamma_a\Psi$, because of Corollary 2, together with the components $p_7 = \Psi^\dagger\Gamma_0\Gamma_7\Psi$ and $p_8 = \Psi^\dagger\Gamma_0i\Psi$ they build up, for Ψ simple spinor of $\mathcal{Cl}_0(7, 1)$, a null vector $P \in \mathbb{R}^{7,1}$ with real components:

$$p_A = \Psi^\dagger\Gamma_0\Gamma_A\Psi \quad \text{where} \quad \Gamma_A = \{\Gamma_a, \Gamma_7, i\mathbb{1}\}. \quad A = 1, \dots, 8. \quad (6.2)$$

They satisfy (identically) to:

$$p_\mu p^\mu + p_5^2 + p_6^2 + p_7^2 + p_8^2 = 0 \quad (6.3)$$

and the corresponding equations for Ψ may be derived from the Cartan's equation:

$$p^A G_A (1 \pm G_9)\Theta = 0 \quad (4.17')$$

with G_A generators of $\mathbb{C}\ell(7,1)$. According to the isomorphisms discussed in Chapter 4, there are four of them (see eq.(4.19)):

$$\left(p^\mu \Gamma_\mu^{(m)} + p_5 \Gamma_5^{(m)} + p_6 \Gamma_6^{(m)} + p_7 \Gamma_7^{(m)} + M \right) \Psi^{(m)} = 0; \quad m = 0, 1, 2, 3, \quad (6.4)$$

depending if $\Psi^{(m)}$ is a doublet of Dirac, Weyl, or Pauli spinors for $m = 0; 1, 2;$ or 3 respectively. Let us suppose $m = 0$ then:

$$\Psi^{(0)} = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} := N$$

where ψ_j are Dirac $\mathbb{C}\ell(3,1)$ -spinors, while Γ_A have the form (see $\Gamma_A^{(0)}$ in eqs.(4.5), (4.7)):

$$\Gamma_\mu^{(0)} = \begin{pmatrix} \gamma_\mu & 0 \\ 0 & \gamma_\mu \end{pmatrix}, \quad \Gamma_5^{(0)} = \begin{pmatrix} 0 & \gamma_5 \\ \gamma_5 & 0 \end{pmatrix}, \quad \Gamma_6^{(0)} = \begin{pmatrix} 0 & -i\gamma_5 \\ i\gamma_5 & 0 \end{pmatrix}, \quad \Gamma_7^{(0)} = \begin{pmatrix} \gamma_5 & 0 \\ 0 & -\gamma_5 \end{pmatrix}.$$

We may now interpret eq.(6.4) as an equation of motion by interpreting p_μ as generators of Poincaré translations and then substitute p_μ with $i\frac{\partial}{\partial x_\mu}$, while defining the spinor N as a function of space-time: $x_\mu \in M = \mathbb{R}^{3,1}$ (in a similar way as Weyl eq.(3.9') became (3.11); in this case $\mathbb{C}\ell(7,1)$ may be interpreted as a fiber over M), and eq.(6.4) becomes:

$$\left(i\frac{\partial}{\partial x_\mu} \cdot 1 \otimes \gamma^\mu + \vec{\pi} \cdot \vec{\sigma} \otimes \gamma_5 + M \right) N = 0 \quad (6.5)$$

where, according to eq.(6.2):

$$p_\mu = \frac{1}{8}(\tilde{\psi}_1 \gamma_\mu \psi_1 + \tilde{\psi}_2 \gamma_\mu \psi_2); \quad \vec{\pi} = \frac{1}{8} \tilde{N} \vec{\sigma} \otimes \gamma_5 N; \quad M = \frac{1}{8} (\tilde{\psi}_1 \psi_1 + \tilde{\psi}_2 \psi_2) \quad (6.6)$$

with $\tilde{N} = (\tilde{\psi}_1, \tilde{\psi}_2)$. With (6.6) eq.(6.3) is an identity provided N is simple or pure [8], and eq.(6.5) is totally spinorial (it may also become an identity).

Eq.(6.5) is formally identical with the well-known proton-neutron equation, when interacting with the pseudo-scalar pion isovector, apart from $M = ip_8$ which here appears to be, like $\vec{\pi}$, x -dependent (the first option of Chapter 3, that is to conceive N as p dependent is here unacceptable since the Fourier transform of the equation of motion would give a non local interaction term).

Here it is seen that p_5, p_6, p_7 in eq.(6.4) represent interaction terms with external fields; in this case the pions, and this will be steadily true, in the following, for all terms after the first four ².

One could also conceive eq.(6.4) as the Fourier transform of an equation of motion in an 8 dimensional Minkowski space $M_8 = \mathbb{R}^{7,1}$ and define N as a field taking

²Eight component spinors are subject to one constraint equation if simple or pure. In this case it could mean $M = 0$ (see eq.(13.1) of ref.[8]) which, in turn, because of eq.(6.5) could imply massless pions by which, notoriously, pion-nucleon physics is greatly simplified: it could then derive from Cartan's simplicity imposed to the nucleon doublet.

values in M_8 , to be after reduced through dimensional reduction; a procedure here avoided by merely generating ordinary space-time through p_μ .

As stated our aim is to determine the equations of motion for fermions conceived as simple or pure spinors where then bosons appear as external fields bilinearly expressed in terms of spinors, as in fact appears from eqs.(6.6). At this point it should be possible to complete the picture by determining the equations of motion for the pion field, bilinearly expressed in terms of the nucleon doublet N as in eq.(6.6), in a similar way as Maxwell's equations may be obtained from the Weyl ones [7],[8]. In fact E. Fermi and C.N. Yang have shown [10] that the pion, conceived as bilinear (composite) in the nucleon field, is apt to represent Yukawa theory.

Observe that the pseudoscalar nature of the pion derives from imposition that N is a doublet of Dirac spinors which, in turn, imposes the representation $\Gamma_a^{(0)}$ of Γ_a where γ_5 must be contained in $\Gamma_5, \Gamma_6, \Gamma_7$ (in order to anticommute with Γ_μ). Again, the fact that the pion field $\vec{\pi}$ appears here as bilinearly expressed in terms of the proton-neutron field does not imply that, in the quantized theory, the pion must be considered as a bound state of proton-neutron.

It is easy to see that if we start from $\mathcal{Cl}(1, 3)$ we obtain $\mathcal{Cl}(1, 7)$ and eq.(6.5) assumes the quaternionic form:

$$(p_\mu \gamma^\mu + s_{4+n} q^n + m)N = 0, \quad n = 1, 2, 3 \quad (6.5')$$

where $q_n = -i\sigma_n$ are the imaginary quaternion units and $m = \frac{1}{8}(\tilde{\psi}_1 \psi_1 + \tilde{\psi}_2 \psi_2)$ is a real scalar (see Appendix A1., eq.(A1, 12''')). The term $s_{4+n} q^n = \vec{\pi} \cdot \vec{\sigma} \otimes \gamma_5$ presents the so-called isospin internal symmetry $SU(2)$ of nuclear forces, here of quaternionic origin.

The group $SU(2)$ is the covering of $SO(3)$ and, traditionally, isotopic spin symmetry was interpreted as a rotational covariance in 3-dimensional isotopic spin space. Here instead it is originated by the generators $\Gamma_5, \Gamma_6, \Gamma_7$ of $\mathcal{Cl}(5, 1)$, or $\mathcal{Cl}(1, 5)$ as they appear in the interaction terms:

$$p_5 \Gamma_5 + p_6 \Gamma_6 + p_7 \Gamma_7 = \vec{\pi} \cdot \vec{\sigma} \otimes \gamma_5 \quad (6.7)$$

of the equation of motion (6.5). Now, as well known, the generators Γ_a of Clifford algebras are reflection operators in spinor space, from which then internal symmetry should arise. And this will be true also in the following [8].

W. Heisenberg, the discoverer of isospin symmetry, hoped to derive it from the conformal group of symmetry, which is notoriously represented by the group $SO(4, 2)$ acting in $\mathbb{R}^{4,2}$, from which however no compact internal symmetry rotational group may be derived. Instead if we accept the indication of our derivation, which suggests the reflection origin of isospin, then, in accordance with Heisenberg's intuition, there is a correlation with the conformal group. In fact if we assume Γ_6 as the second time-like generator of $\mathcal{Cl}(4, 2)$, besides Γ_0 , the generators of the extra space-time reflections of the conformal group are:

$$\Gamma_5, i\Gamma_6, \Gamma_7 \quad (6.8)$$

(where the imaginary unit i in front of Γ_6 is due to the fact that the square of a reflection must be equal to the identity) which are in fact identical (apart from the

factor γ_5) to the generators appearing in eq.(6.5) or (6.7). Therefore we may affirm that, according to our derivation, isospin internal symmetry $SU(2)$ originates from reflections which identify with those of the conformal group in turn correlated with the imaginary units of the quaternion field of numbers. This correlation with conformal reflections might have further interesting and far reaching implications: see ref.[8].

6.2 Electric charge

Also the possible origin of electric charge might appear in this third step. In fact let us write down explicitly eq. (6.5) in terms of the two Dirac spinors ψ_1 and ψ_2 of the N doublet:

$$\begin{aligned} (p_\mu \gamma_\mu + p_7 \gamma_5 + i p_8) \psi_1 + \gamma_5 (p_5 - i p_6) \psi_2 &= 0, \\ (p_\mu \gamma_\mu - p_7 \gamma_5 + i p_8) \psi_2 + \gamma_5 (p_5 + i p_6) \psi_1 &= 0. \end{aligned} \quad (6.9)$$

Now all p_a are real, therefore defining

$$p_5 \pm i p_6 = \rho e^{\pm i \frac{\omega}{2}}$$

and multiplying the first eq.(6.9) by $e^{i \frac{\omega}{2}}$ we obtain

$$\begin{aligned} (p_\mu \gamma_\mu + p_7 \gamma_5 + i p_8) e^{i \frac{\omega}{2}} \psi_1 + \gamma_5 \rho \psi_2 &= 0, \\ (p_\mu \gamma_\mu - p_7 \gamma_5 + i p_8) \psi_2 + \gamma_5 \rho e^{i \frac{\omega}{2}} \psi_1 &= 0. \end{aligned} \quad (6.10)$$

We see then that ψ_1 appears with a phase factor $e^{i \frac{\omega}{2}}$ corresponding to a rotation through an angle ω in the circle defined by

$$p_5^2 + p_6^2 = \rho^2 \quad (6.11)$$

in the vector space of the Klein quadric defined by eq.(6.3), which in turn corresponds to an imaginary dilation in $\mathbb{R}^{4,2}$. In fact the corresponding transformation in spinor space is generated by the Lie algebra element

$$J_{56} = \frac{1}{2} [\Gamma_5, \Gamma_6] \quad (6.12)$$

which is obtained from J_{56} in the $SU(2, 2)$ covering of the conformal group, after multiplying the generation Γ_6 by the imaginary unit i .

Observe that this complexification was intrinsic to the modality of our construction which brought us to $\mathcal{Cl}(5, 1)$ which may be obtained by setting an imaginary unit factor i in front of the generator Γ_6 of $\mathcal{Cl}(4, 2)$ as in (6.8) and which finally generated the $SU(2)$ internal symmetry.

With this interpretation the (non compact) dilation covariance of the conformal group induced the (compact) $U(1)$ group of symmetry represented by the phase factor in front of ψ_1 and not of ψ_2 (or vice-versa).

As stated above we may translate this in the corresponding Minkowski space-time, and since dilatation covariance is local, we may consider the phase angle ω as coordinate dependent $\omega \rightarrow \omega(x)$ and then to maintain the covariance of eq.(6.9)

we will have to introduce an abelian gauge potential A_μ interacting with ψ_1 only and, as easily seen, the eq.(6.5) will become, in space-time $\mathbb{R}^{3,1}$:

$$\left\{ \gamma_\mu \left[i \frac{\partial}{\partial x_\mu} + \frac{e}{2} (1 - i\Gamma_5 \Gamma_6) A_\mu \right] + \vec{\pi} \cdot \vec{\sigma} \otimes \gamma_5 + M \right\} \begin{pmatrix} p \\ n \end{pmatrix} = 0 \quad (6.13)$$

well representing the equation of the proton-neutron doublet interacting with the pion and with the electromagnetic potential A_μ determining $F_{\mu\nu}$, satisfying eqs.(3.14) in empty space.

Observe that the electric charge Q of the nucleon doublet N is then represented by:

$$Q = \frac{e}{2} (1 - i\Gamma_5 \Gamma_6), \quad (6.14)$$

where $-i\Gamma_5 \Gamma_6 = \sigma_3 \otimes 1$, that is the third component of isospin generator. Furthermore the proton p and the neutron n are eigenstates of $-i\Gamma_5 \Gamma_6$ corresponding to the eigenvalues $+1$ and -1 , respectively (see also ref.[8]).

Quaternions might have a role also in the origin of the electroweak model where also a doublet of (left-handed) fermions appears, as we will see in the next section.

6.3 The electroweak model

Let us go back to eq.(6.4) representing four equivalent equations of motion for the fermion doublet $\Psi^{(m)}$ labelled by the index $m = 0, 1, 2, 3$. In section 6.1 we adopted $m = 0$ corresponding to a doublet of Dirac $\mathbb{C}\ell(3, 1)$ spinors, obtaining $SU(2)$ symmetric pion-nuclear equation (6.5). For $m = 1, 2, 3$, corresponding to Weyl and Pauli doublets, we may expect to obtain another $SU(2)$ symmetric equation: it will be the one of the electroweak model.

In fact the corresponding spinors $\Psi^{(m)}$ are isomorphic, however, for what is observed in Chapter 4, they are not physically equivalent. Suppose now that the doublet $\Psi^{(m)}$ represents a lepton pair whose electroweak charge derives from a local $U(1)$ phase covariance of $\Psi^{(m)}$, (as we will see in Chapter 7). Then a minimal coupling (or covariant derivative in space time) will impose a gauge vector field $A_\mu^{(m)}$ and (6.4) will become:

$$\left[(p^\mu - A_\mu^{(m)}) \Gamma_\mu^{(m)} + \sum_{j=5}^7 p_j \Gamma_j^{(m)} + M \right] \Psi^{(m)} = 0, \quad m = 0, 1, 2, 3 \quad (6.15)$$

where:

$$A_\mu^{(m)} = \tilde{\Psi}^{(m)} \Gamma_\mu^{(m)} \Psi^{(m)}. \quad (6.16)$$

Let us now consider the unitary operators U_j defined in eqs.(4.9) where now L and R are the chiral projectors:

$$L = \frac{1}{2}(1 + \gamma_5); \quad R = \frac{1}{2}(1 - \gamma_5). \quad (6.17)$$

The operators U_j represent the isomorphisms of eqs.(4.4) because of which:

$$U_j \Psi^{(j)} = \Psi^{(0)}; \quad j = 1, 2, 3. \quad (6.18)$$

Let us now indicate with $\Psi_L^{(m)}$ the left-handed part of $\Psi^{(m)}$:

$$\Psi_L^{(m)} = 1_2 \otimes L \Psi^{(m)} = \begin{pmatrix} \psi_{1L} \\ \psi_{2L} \end{pmatrix} \quad (6.19)$$

then, since the U_j , given in (4.9), have in common the term $1_2 \otimes L$, as a consequence of (6.18), we have that

$$\Psi_L^{(m)} = \Psi_L^{(m')} \quad (6.20)$$

for all four values of m and m' .

Now since $\Psi^{(m)} = \Psi_L^{(m)} + \Psi_R^{(m)}$ we have that, summing eqs.(6.15) for the indices $m = 1, 2, 3$ they will give rise, reminding that $\Gamma_\mu^{(j)} = \sigma_j \otimes \gamma_\mu$ (see eqs.(4.5)), to the equation:

$$\sum_{k=1}^3 \left(p^\mu \Gamma_\mu^{(k)} + \sum_{j=1}^7 p_j \Gamma_j^{(k)} + M \right) \Psi^{(k)} - \sum_{k=1}^3 A_{(k)}^\mu \Gamma_\mu^{(k)} \Psi_R^{(k)} - \vec{A}^\mu \cdot \vec{\sigma} \otimes \gamma_\mu \Psi_L^{(0)} = 0, \quad (6.21)$$

where the last term presents an $SU(2)_L$ internal gauge symmetry (this time in the dynamical sector of the equation).

If we suppose that the lepton doublet $\Psi^{(0)}$ represents an electron e and a massless, left-handed neutrino ν_L :

$$\Psi^{(0)} = \begin{pmatrix} e \\ \nu_L \end{pmatrix},$$

we will have that, because of eq.(6.20):

$$\Psi_L^{(m)} = \Psi_L^{(0)} = \begin{pmatrix} e_L \\ \nu_L \end{pmatrix}; \quad \Psi_R^{(m)} = e_R, \quad m = 0, 1, 2, 3 \quad (6.22)$$

and, after summing eqs.(6.15), taking into account that, for leptons, not possessing strong charges, the terms p_5, p_6, p_7 will be absent, we easily obtain:

$$(p_\mu \gamma^\mu + M) \begin{pmatrix} e \\ \nu_L \end{pmatrix} - \vec{A}_\mu \cdot \vec{\sigma} \times \gamma^\mu \begin{pmatrix} e_L \\ \nu_L \end{pmatrix} + (B_\mu \gamma^\mu + \tau) e_R = 0 \quad (6.23)$$

where:

$$\vec{A}_\mu = \tilde{\Psi}^{(0)} \vec{\sigma} \otimes \gamma_\mu \Psi^{(0)},$$

B_μ is an isosinglet four-vector and τ a scalar. Eq.(6.23) represents the geometrical starting point for the electroweak model.

6.4 Dirac's equation

Both from eq.(6.5) and from eq.(6.23) we may obtain the free Dirac's equation; in fact we need only to eliminate from (6.5) the pion interaction terms and what remains is the Dirac's equation for the proton and the neutron; similarly from (6.23) eliminating the electroweak interactions we obtain the Dirac's equation for the free electron. We see then that the free Dirac equation is only conceivable as an approximate equation when interactions may be ignored, which is natural since the Dirac spinor is complex. The only exact equation of motion, for four component spinors which naturally emerges in our construction is the Majorana one: eq.(5.3).

7 FOURTH STEP: THE STRONG (OR ELECTROWEAK) CHARGE

7.1 The baryon-lepton quadruplet

The construction may be continued observing that the real null vector with components p_A given by eq. (6.2) for N , thought as a Weyl spinor of $\mathbb{C}\ell_0(7, 1)$, may be considered as a particular case of the following:

$$P_A^\pm = \Theta^\dagger G_0 G_A (1 \pm G_9) \Theta, \quad A = 1, 2, \dots, 8, \quad (7.1)$$

where Θ is a sixteen component spinor associated with $\mathbb{C}\ell(7, 1)$ of which G_A are the generators and G_9 the volume element.

Again Θ may be considered in the Dirac basis

$$\Theta = \begin{pmatrix} N_1 \\ N_2 \end{pmatrix} \quad (7.2)$$

where N_1 and N_2 are Dirac spinors of $\mathbb{C}\ell(5, 1)$ and, again, defining

$$P_A = P_A^+ + P_A^- = \Theta^\dagger G_0 G_A \Theta \quad A = 1, 2, \dots, 8 \quad (7.3)$$

and

$$\begin{aligned} P_9 &= \Theta^\dagger G_0 G_9 \Theta \\ P_{10} &= i\Theta^\dagger G_0 \Theta \end{aligned} \quad (7.4)$$

we obtain, for Θ simple or pure, the components P_α of a ten dimensional real null vector which defines the Cartan's equation for the spinor Φ , associated with $\mathbb{C}\ell(9, 1)$, with generators \mathcal{G}_α :

$$P_\alpha \mathcal{G}_\alpha (1 \pm \mathcal{G}_{11}) \Phi = 0 \quad \alpha = 1, 2, \dots, 10 \quad (4.17'')$$

from which the equation for the 16-component spinor Θ may be derived:

$$(P_a G^a + P_7 G_7 + P_8 G_8 + P_9 G_9 + iP_{10}) \Theta = 0, \quad (7.5)$$

where $a = 1, 2 \dots 6$. For Θ in the Dirac basis we may, with the same procedure as that of section 6.2, show that the Dirac spinors N_1 and N_2 obey a system of equations where N_1 (or N_2) is multiplied by a phase factor $e^{i\tau/2}$, where the angle τ represents a rotation in the circle

$$P_7^2 + P_8^2 = \rho^2 \quad (7.6)$$

for which eq. (7.5) is covariant (in spinor space it is generated by G_7G_8). This $U(1)$ symmetry of N_1 (or N_2) may be interpreted as a charge which, being different from the electric charge (generated by $\Gamma_5\Gamma_6$), could be the charge of strong forces. In this case then, $N_1 = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}$ could represent the nucleon doublet while $N_2 = \begin{pmatrix} \psi_3 \\ \psi_4 \end{pmatrix}$ the electron neutrino doublet say, both of which contain an electrically charged and neutral component. Then it is easy to see that, since N_1 and N_2 are eigenstates of $-iG_7G_8$ corresponding to the eigenvalues $+1$ and -1 respectively, while the Dirac spinor components of N_1 and N_2 are eigenstates of $-i\Gamma_5\Gamma_6$, again corresponding to the eigenvalues $+1, -1$, if the charged partner of the nucleon doublet N_1 (the proton) has the charge $+e$, the charged partner of the lepton doublet N_2 (the electron) should have the charge $-e$ (see ref.[8]), as it happens, in fact.

As seen above in each of the doublets quaternions generate an internal symmetry which could then be $SU(2)$ isospin for N_1 and $SU(2)_L$ electroweak for N_2 . Suppose now that, as discussed in section 6.2, the rotation in the circle (7.6) generating $U(1)$ is local, then it will generate a gauge interaction, like it did in eq.(6.13); however now, because of those $SU(2)$ it will be a non abelian gauge field theory (and this time, despite the reflection origin in spinor space, those $SU(2)$ may be conceived as covering of rotation groups, as will be discussed elsewhere), obviously to be completed, with the dynamical part of Yang-Mills.

8 THE FIFTH AND LAST STEP

Continuing the construction, the P_α of eqs.(7.3) and (7.4) are a particular realization of the following:

$$P_\alpha^\pm = \Phi^\dagger \mathcal{G}_0 \mathcal{G}_\alpha (1 \pm \mathcal{G}_{11}) \Phi \quad \alpha = 1, 2, \dots, 10. \quad (8.1)$$

Then, for Φ simple (as Weyl of $\mathbb{C}\ell_0(11, 1)$)

$$P_\alpha = P_\alpha^+ + P_\alpha^- = \Phi^\dagger \mathcal{G}_0 \mathcal{G}_\alpha \Phi \quad (8.2)$$

together with

$$P_{11} = \Phi^\dagger \mathcal{G}_0 \mathcal{G}_{11} \Phi \quad P_{12} = i\Phi^\dagger \mathcal{G}_0 \Phi \quad (8.3)$$

build up the components of a 12 dimensional real null vector defining the Cartan's equation

$$(P_A \mathcal{G}^A + P_9 \mathcal{G}_9 + P_{10} \mathcal{G}_{10} + P_{11} \mathcal{G}_{11} + iP_{12}) \Phi = 0, \quad A = 1, 2, \dots, 8. \quad (8.4)$$

For Φ , in the Dirac spinor representation

$$\Phi = \begin{pmatrix} \Theta_1 \\ \Theta_2 \end{pmatrix} \quad (8.5)$$

where Θ_1 and Θ_2 may be considered as Dirac spinors of $\mathbb{C}\ell(7, 1)$, eq.(8.4) will present, with the same procedure as that of section 6.2 and Chapter 7, an $U(1)$ phase invariance of Θ_1 generated by $\mathcal{G}_9\mathcal{G}_{10}$, corresponding to a rotation through an angle σ in the circle $P_9^2 + P_{10}^2 = \rho^2$. We may interpret it as the $U(1)$ corresponding to a strong charge (or hypercharge). Then the quadruplet of fermions contained in Θ_1 and Θ_2 could represent baryons and leptons respectively, and each one of them obeys in principle to an equation like (7.5).

Clifford algebras $\mathbb{C}\ell(k, l)$ obey the notorious periodicity theorem:

$$\mathbb{C}\ell(l + 4, k + 4) = \mathbb{C}\ell(l + 8, k) = \mathbb{C}\ell(l, k + 8) = \mathbb{C}\ell(l, k) \otimes \mathcal{R}(16) \quad (8.6)$$

where $\mathcal{R}(16)$ stands for the algebra of 16×16 real matrices. Therefore, the maximal Clifford algebra to study in our construction will be the algebra

$$\mathbb{C}\ell(9, 1) = \mathcal{R}(32) = \mathbb{C}\ell(1, 9), \quad (8.7)$$

admitting real Majorana-Weyl spinors, since after this the geometrical structures, because of the periodicity theorem, will repeat themselves, and, furthermore the number of constraint equations becomes too high: 66 and 364 for $\mathbb{C}\ell(11, 1)$ and $\mathbb{C}\ell(13, 1)$, respectively.

Before examining the possibility of representing with Θ_1 and Θ_2 baryons and leptons respectively, we wish first to define dimensional reduction procedure in our formulation and to study its possible physical meaning.

9 DIMENSIONAL REDUCTION

In our approach, dimensional reduction will simply consist in reversing the steps of our construction. Precisely if Φ represents a 2^n component Dirac spinor associated to the Clifford algebra $\mathbb{C}\ell(2n - 1, 1)$, with generators $\gamma_a : \{\gamma_1, \gamma_2, \dots, \gamma_{2n}\}$ we may reduce the spinor Φ to the 2^{n-1} component Weyl spinors φ_{\pm}^W through the projectors $\pi_{\pm}^{(1,2)} = \frac{1}{2}(1 \pm \gamma_{2n+1})$ see [8]:

$$\pi_{\pm}^{(1,2)} : \Phi \rightarrow \frac{1}{2}(1 \pm \gamma_{2n+1})\Phi = \varphi_{\pm}^W. \quad (9.1)$$

Correspondingly the $2n + 2$ dimensional null vector with components:

$$P_A = \Phi^\dagger \gamma_0 \gamma_A \Phi \quad A = 1, 2, \dots, 2n + 2 \quad (9.2)$$

where $\gamma_A : \{\gamma_a, \gamma_{2n+1}, i\mathbb{1}\}$ will be reduced to:

$$\pi_{\pm}^{(1,2)} : P_A \rightarrow p_a^{\pm} = \Phi^\dagger \gamma_0 \gamma_a (1 \pm \gamma_{2n+1}) \Phi, \quad a = 1, 2, \dots, 2n \quad (9.3)$$

since, according to Corollary 2, $P_{2n+1} \equiv 0 \equiv P_{2n+2}$.

However there are also other possibilities. In fact according to the isomorphisms indicated in (4.4) we may also reduce the Dirac spinor Φ to 2^{n-1} components Dirac spinors φ_{\pm}^D associated with the Clifford algebra $\mathbb{C}\ell(2n-3, 1)$. It is easy to see [8] that in this case we have to use the projectors $\pi_{\pm}^{(0)} = \frac{1}{2}(1 \pm i\gamma_{2n-1}\gamma_{2n})$:

$$\pi_{\pm}^{(0)} : \Phi \rightarrow \frac{1}{2}(1 \pm i\gamma_{2n-1}\gamma_{2n})\Phi = \varphi_{\pm}^D \quad (9.4)$$

and correspondingly we have $P_{2n-1} \equiv 0 \equiv P_{2n}$; that is

$$\pi_{\pm}^{(0)} : P_A \rightarrow p_b^{\pm} = \Phi^{\dagger}\gamma_0\gamma_b(1 \pm i\gamma_{2n-1}\gamma_{2n})\Phi \quad (9.5)$$

where $b = 1, 2, \dots, 2n-2, 2n+1$.

According to the isomorphisms indicated in (4.4) we may also reduce the spinor Φ to Pauli spinors associated with $\mathbb{C}\ell(2n-2, 1)$. It is easy to see [8] that in this case we have to use the projectors: $\pi_{\pm}^{(3)} = \frac{1}{2}(1 \pm i\gamma_{2n}\gamma_{2n+1})$:

$$\pi_{\pm}^{(3)} : \Phi \rightarrow \frac{1}{2}(1 \pm i\gamma_{2n}\gamma_{2n+1})\Phi = \varphi_{\pm}^P \quad (9.6)$$

as a consequence of which: $P_{2n} \equiv 0 \equiv P_{2n+1}$.

We see then that there are more possibilities of dimensional reduction each of which halves the dimension of spinor space and reduces by two the dimension of momentum space. The reduced spinor spaces are isomorphic because of the isomorphisms represented in (4.4), (4.11) while, what changes in each reduction, are the two components of momentum space, appearing in the interaction terms, which are eliminated from the equations of motion (see also ref.[8]), therefore the reduced spinors may result physically not equivalent, and this inequivalence in fact may give rise to the existence of families as we will see in the next Chapter.

Dimensional reduction will eliminate P_A with $A \geq 5$ and then eliminate the interaction terms from the equation of motion, which in turn will mean descending from higher to lower energy phenomena. P_{μ} with $\mu = 1, 2, 3, 0$ will represent the dynamical term, which when interpreted as Poincaré translation, generate space-time, which in this approach is, and remains, four dimensional.

10 BARYONS AND LEPTONS

10.1 Baryons

Let us now assume in eq.(8.5) the 16 component spinor Θ_1 to be of the form:

$$\Theta_1 = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{pmatrix} := \Theta_B \quad (10.1)$$

where b_j are $\mathbb{C}\ell(3,1)$ -Dirac spinors, representing a quadruplets of fermions or quarks, presenting strong charge represented by its $U(1)$ covariance generated

by $\mathcal{G}_9\mathcal{G}_{10}$. Suppose it obeys eq.(7.5) with P_α given by (7.3), (7.4), being the components of a null vector satisfy identically to:

$$-P_\mu P^\mu = P_5^2 + P_6^2 + P_7^2 + P_8^2 + P_9^2 + P_{10}^2 = \mathcal{M}^2, \quad (10.2)$$

where \mathcal{M} is an invariant mass. The directions of P_5, P_6, \dots, P_{10} define S^5 . Therefore it is to be expected that the quadruplets may present a maximal $SU(4)$ internal symmetry (covering group of $SO(6)$), which will be the obvious candidate for the maximal possible flavor internal symmetry. The most straightforward way to set it in evidence is to determine the 15 generators if the Lie algebra of $SU(4)$ (determined by the commutators $[G_i, G_k]$ for $5 \leq j, k \leq 10$) represented by 4×4 matrices, whose elements are either 1_4 or γ_5 , acting in the space of the Θ_B quadruplet. Let them be λ_j ; where $1 \leq j \leq 15$. Denote with f_j , $1 \leq j \leq 15$ the components of an emisymmetric tensor building up an automorphism space of $SO(6)$. Then, a natural³ equation of motion for Θ_B could have the general form:

$$\left(P_\mu G^\mu + \sum_{j=1}^3 \lambda_j f^j + \sum_{j=4}^8 \lambda_j f^j + \sum_{j=9}^{15} \lambda_j f^j \right) \Theta_B = 0 \quad (10.3)$$

where $\lambda_1 \lambda_2 \lambda_3$ are the generators of $SU(2)$; $\lambda_1, \dots, \lambda_8$ those of $SU(3)$ and $\lambda_1, \dots, \lambda_{15}$ those of $SU(4)$.

In order to study the possible physical information contained in the quadruplet Θ_B , let us now operate with our dimensional reduction.

The most obvious will be to adopt the projector $\pi_\pm^{(1,2)} = \frac{1}{2}(1 \pm G_9)$ which is the image in $\mathcal{Cl}(9, 1)$ of $\frac{1}{2}(1 \pm \mathcal{G}_9\mathcal{G}_{10})$ of $\mathcal{Cl}(1, 11)$, and, as we have seen in Chapter 8, $\mathcal{G}_9\mathcal{G}_{10}$ was the generator of $U(1)$ corresponding to the strong charge. Now, the reduction

$$\Theta_B \rightarrow \frac{1}{2}(1 \pm G_9) \Theta_B = N_\pm \quad (10.4)$$

implies that

$$p_{9,10}^\pm = \Theta_B^\dagger G_0 G_{9,10} (1 \pm G_9) \Theta_B \equiv 0. \quad (10.5)$$

Therefore the null vector P_α given by (7.3) and (7.4) reduces to p_A given by eq.(6.2), which means that N_+ or N_- , conceived as a doublet of fermions obeys eq.(6.5) of the nucleon doublet say, presenting an $SU(2)$ isospin internal symmetry generated by quaternions. In this way we may realize the first term, after the dynamical one $P_\mu G^\mu$, in eq.(10.3). But we know that at least also the second one representing $SU(3)$ symmetry should be possible. In the next Chapter we will see that it may emerge from octonions.

³In a similar way as $(i\partial_\mu \gamma^\mu + [\gamma_\mu, \gamma_\nu] F^{\mu\nu} + m) \psi = 0$, where $F_{\mu\nu}$ is the electromagnetic tensor, is a natural equation for a fermion, the neutron say, presenting an anomalous magnetic moment.

10.2 Leptons and families

We will now interpret the 16-component spinors Θ_2 in eq.(8.5) as a quadruplet of leptons:

$$\Theta_2 := \begin{pmatrix} \ell_1 \\ \ell_2 \\ \ell_3 \\ \ell_4 \end{pmatrix} := \Theta_{\mathcal{L}}. \quad (10.1')$$

We may now operate the dimensional reduction of $\Theta_{\mathcal{L}}$ with the projector $\pi_{\pm}^{(1,2)} = \frac{1}{2}(1 \pm G_9)$ as we did for Θ_B :

$$\Theta_{\mathcal{L}} \rightarrow \frac{1}{2}(1 \pm G_9)\Theta_{\mathcal{L}} = \varphi_{\pm}^W \quad (10.4')$$

and consequently $p_9^{\pm} \equiv 0 \equiv p_{10}^{\pm}$. Now, at difference with the baryon quadruplet, for leptons, dimensional reduction, equivalent to decoupling, is necessary, since the leptons, not presenting strong change, may neither obey eq.(7.5) nor to the full eq.(10.3); and then the doublet φ_{\pm}^W say, may only present the $SU(2)_L$ symmetry of the electroweak model, as shown in section 6.3.

The eight component spinors φ_{\pm}^W are Weyl spinors and therefore the first four generators G_{μ} of $\mathbb{C}\ell(7, 1)$, which determines their physical behaviour, or transformation properties with respect to the Poincaré group will be:

$$G_{\mu}^{(1,2)} = \sigma_{1,2} \otimes \Gamma_{\mu} \quad (10.6)$$

where Γ_{μ} are the first four generators of $\mathbb{C}\ell(5, 1)$. These in turn, which determine the physical behaviour of the fermions in the doublet, will have, according to eqs.(4.5), the following possible forms:

$$\Gamma_{\mu}^{(1,2)} = \sigma_{1,2} \otimes \gamma_{\mu}, \quad \Gamma_{\mu}^{(3)} = \sigma_3 \otimes \gamma^{\mu} \quad \text{or} \quad \Gamma_{\mu}^{(0)} = \mathbb{1} \otimes \gamma_{\mu} \quad (10.6')$$

depending if we have a doublet of Weyl, Pauli or Dirac spinors respectively.

We may also conceive $\Theta_{\mathcal{L}}$ as a doublet of Pauli spinors, and consequently reduce it with the projector $\pi_{\pm}^{(3)} = \frac{1}{2}(1 \pm iG_8G_9)$:

$$\Theta_{\mathcal{L}} \rightarrow \frac{1}{2}(1 \pm iG_8G_9)\Theta_{\mathcal{L}} = \varphi_{\pm}^P \quad (10.4'')$$

because of which $p_8^{\pm} \equiv 0 \equiv p_9^{\pm}$ and the generators G_{μ} will be:

$$G_{\mu}^{(3)} = \sigma_3 \otimes \Gamma_{\mu} \quad (10.7)$$

and, again Γ_{μ} may have the forms (10.6').

For $\Theta_{\mathcal{L}}$ doublet of Dirac spinors we will adopt the projector $\pi_{\pm}^{(0)} = \frac{1}{2}(1 \pm iG_7G_8)$:

$$\Theta_{\mathcal{L}} \rightarrow \frac{1}{2}(1 \pm iG_7G_8)\Theta_{\mathcal{L}} = \varphi_{\pm}^D \quad (10.4''')$$

and then: $p_7^{\pm} \equiv 0 \equiv p_8^{\pm}$ while:

$$G_{\mu}^{(0)} = \mathbb{1} \otimes \Gamma_{\mu} \quad (10.8)$$

and again Γ_μ may have the forms (10.6').

We see then through the projectors $\pi_\pm^{(1,2)}, \pi_\pm^{(3)}, \pi_\pm^{(0)}$, there is the possibility of reduction of $\Theta_{\mathcal{L}}$ to an 8-component Weyl, Pauli or Dirac spinor respectively (correlated with quaternions) which are isomorphic according to (4.4) and (4.11). Not equivalent are instead the corresponding dimensional reduction in momentum space since the eliminated components of momentum will be: $p_{10}, p_9; p_9 p_8$ and p_8, p_7 for the Weyl, Pauli and Dirac case respectively. Now, since these components appear in the equations of motion as interaction terms, the three reduced lepton doublets will obey different equations of motion obtained from eq.(7.5) eliminating the corresponding pair of terms. Correspondingly the invariant mass equation (10.2) will reduce to:

$$\begin{aligned} -p_\mu p^\mu &= p_5^2 + p_6^2 + p_7^2 + p_8^2 = m_W^2 \\ -p_\mu p^\mu &= p_5^2 + p_6^2 + p_7^2 + p_{10}^2 = m_P^2 \\ -p_\mu p^\mu &= p_5^2 + p_6^2 + p_9^2 + p_{10}^2 = m_D^2 \end{aligned} \quad (10.9)$$

for the Weyl, Pauli and Dirac case respectively.

Dimensional reduction of the lepton quadruplet brings us then to 3 fermion doublets of charged-neutral leptons with different invariant masses m_W, m_P, m_D , each one of them lower than the one M of eq.(10.2) of the baryon multiplet, reminding the 3 families of leptons discovered in nature.

In our imbedding of spinors and compact momentum spaces in higher dimensional ones we could represent physical phenomena whose characteristic energy and momentum steadily increased with the dimension of the spaces. One could then expect that the mean values of the components with higher indices (9,10) should be higher than those with lower ones (5,6). In this expectation then the e, μ, τ leptons could then be identified with $\varphi^W, \varphi^P, \varphi^D$ respectively.

We could obviously repeat the same reductions with the baryon quadruplet Θ_B and we would obtain 3 families of quarks to be correlated with the corresponding 3 families of leptons, and with similar properties, as it effectively appears in natural phenomena.

The lepton quadruplet $\Theta_{\mathcal{L}}$ presents a further problem. In fact we know that because of the absence of strong charge it has to be reduced to the doublets φ_\pm given in (10.4'), (10.4'') and (10.4''') and we also know that $G_7 G_8$ generates an $U(1)$ phase invariance for φ_+ say, which could be interpreted as the electroweak charge. But then the fermion doublets represented by φ_- should be free from it. Their fermions could then obey to eq.(5.3) (or eq.(5.4)) that is they could be Majorana spinors, only subject to gravitational forces. Furthermore if we trust the above picture, they should be abundant in nature as leptons are. What could then come naturally to mind is dark matter.

Baryons present notoriously, besides the internal symmetry $SU(2)$ possibly of quaternionic origin, also the internal symmetry $SU(3)$ both of flavour and of color. We will show that it might originate from the third and last division algebra; that of octonions.

11 OCTONIONS

11.1 In Clifford algebras

The Clifford algebra $\mathbb{C}\ell(9, 1)$ or $\mathbb{C}\ell(1, 9)$ may be represented with octonions; in fact it is known [11] that:

$$\text{Spin}(9, 1) = \text{Spin}(1, 9) = SL(2, \mathbf{o}) = SL(32, \mathbb{R}) \quad (11.1)$$

where \mathbf{o} stands for octonions (see Appendix A2). Independently, but coherently with our approach, a spinor equation of motion in a ten dimensional momentum space $\mathbb{R}^{1,9}$ was proposed by T. Dray and C.A. Manogue [12] of the form:

$$P\Theta := \begin{pmatrix} p_{10} + p_9 & p_8 - \sum_{j=1}^7 p_j e_j \\ p_8 + \sum_{j=1}^7 p_j e_j & p_{10} - p_9 \end{pmatrix} \begin{pmatrix} \mathbf{o}_1 \\ \mathbf{o}_2 \end{pmatrix} = 0 \quad (11.2)$$

where e_j are the octonion units and $\mathbf{o}_1, \mathbf{o}_2$ octonions.

Let us now write Cartan's eq.(7.5), for the baryon and lepton quadruplets Θ_B and Θ_L discussed in the previous chapter, in matrix form. Adopt for the generators G_A of $\mathbb{C}\ell(7, 1)$ the representation:

$$G_a = \sigma_2 \otimes \Gamma_a; \quad G_7 = \sigma_2 \otimes \Gamma_7, \quad G_8 = \sigma_1 \otimes 1, \quad G_9 = \sigma_3 \otimes 1, \quad a = 1, 2, \dots, 6$$

and we obtain:

$$P\Theta_{\mathcal{L}} = \begin{pmatrix} \pm i p_{10} + p_9 & p_8 - i \sum_{j=1}^7 p_j \Gamma_j \\ p_8 + i \sum_{j=1}^7 p_j \Gamma_j & \pm i p_{10} - p_9 \end{pmatrix} \begin{pmatrix} \varphi_+^W \\ \varphi_-^W \end{pmatrix} = 0. \quad (11.2')$$

Comparing it with (11.2) we see that φ_+^W, φ_-^W substitute the octonions $\mathbf{o}_1, \mathbf{o}_2$ and the seven generators $i\Gamma_j$ the octonion units e_j . The imaginary unit i in front of p_{10} simply means that while in (11.2) p_{10} is the energy or time-like direction, in (11.2') it is p_0 . The $+$ and $-$ sign in front of p_{10} refer to the Baryon and Lepton multiplets respectively (equivalent forms of eq.(11.2') are given in Appendix A1 for the signature 1, 9).

Eqs.(11.2) and (11.2') should be equivalent because of eqs.(11.1), and in fact also in ref.[12], through reduction from 10- to 4-dimensional momentum space, the equations of motion for three families of lepton - massless neutrino pairs are obtained as correlated with the quaternion units.

The privilege of eqs.(11.2') is that their physical meaning is transparent (one could easily transform them in Minkowski space-time and then build up there the corresponding familiar Lagrangian formalism). With eq.(11.2) instead one may hope to discover the possible role in physics of octonions after the one of quaternions. In fact the group of automorphism G_2 of octonions has a subgroup $SU(3)$, once a preferred direction, or octonion unit, is selected. However with eq.(11.2) one has to pay the price of non associativity of the algebra of octonions. To overcome this difficulty a matrix representation of octonions could be helpful in order to deal with them in Clifford algebras. We will try to find one in the frame of $\mathbb{C}\ell(2, 3)$; the anti De Sitter Clifford algebra.

Let us in fact assume

$$\gamma_n = \begin{pmatrix} 0 & -\sigma_n \\ \sigma_n & 0 \end{pmatrix}; \quad \gamma_0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \gamma_5 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad (11.3)$$

where $n = 1, 2, 3$, as generators of $\mathbb{C}\ell(2, 3)$. Then, if e_1, e_2, \dots, e_7 are the seven imaginary units of octonions and e_0 the unit of the algebra, we define, following J. Daboul and R. Delbourgo [13]:

$$e_0 := 1_4, \quad e_n := \gamma_n; \quad e_7 := i\gamma_5; \quad \hat{e}_n := e_{n+3} := e_n e_7 = i\gamma_n \gamma_5. \quad (11.4)$$

In ref.[13] it is shown that (11.4) close the octonion algebra after an appropriate modification of the rule of matrix multiplication. We will instead indicate with the symbol \odot the product of the first six octonion units defined by

$$e_j \odot e_k := \delta_{jk} e_j e_k + (1 - \delta_{jk}) i\gamma_5 \gamma_0 e_j \gamma_0 e_k \gamma_0 \quad (11.5)$$

where $j, k = 1, 2, \dots, 6$, and, on the r.h.s. all products mean standard matrix multiplication, then (11.5) taking into account of (11.4) which defines the products with the seventh, e_7 , close the octonion algebra (see A2).

11.2 $SU(3)$ flavor

Let us now consider $\mathbb{C}\ell(1, 7)$ and the corresponding Cartan's equation:

$$P_A G^A (1 \pm G_9) \Theta = 0 \quad A = 1, 2, \dots, 8 \quad (11.6)$$

and adopt for G_A the representation (see (A1.10)):

$$G_A^{(2)} : \{G_a = \sigma_2 \otimes \Gamma_a; \quad G_7 = -i\sigma_2 \otimes \Gamma_7, \quad G_8 = -i\sigma_1 \otimes 1_8, \quad G_9 = \sigma_3 \otimes 1_8\}, \quad (11.7)$$

where $a = 1, 2, \dots, 6$; then, if Γ_a and Γ_7 are assumed in the Dirac representation $\Gamma_a^{(0)}$ and $\Gamma_7^{(0)}$ (see A1.7) we have:

$$G_{5,6,7} = \begin{pmatrix} 0 & -\sigma_{1,2,3} \\ \sigma_{1,2,3} & 0 \end{pmatrix} \otimes \gamma_5 = -i\sigma_2 \otimes \sigma_{1,2,3} \otimes \gamma_5. \quad (11.8)$$

Therefore we may define:

$$G_{4+n} := e_n \otimes \gamma_5; \quad iG_9 := e_7 \otimes 1_4; \quad iG_{4+n} G_9 := \hat{e}_n \otimes \gamma_5. \quad (11.9)$$

In this way we get a definition of the octonion unit elements in $\mathbb{C}\ell(1, 7)$ which with the \odot product close the octonion algebra (see A2).

Similarly for $\mathbb{C}\ell(1, 9)$ with generators \mathcal{G}_α and Cartan's equation

$$P_\alpha \mathcal{G}^\alpha (1 \pm \mathcal{G}_{11}) \Phi = 0 \quad \alpha = 1, 2, \dots, 10 \quad (11.10)$$

we find that (see $\mathcal{G}_A^{(2)}$ in A1.15):

$$\mathcal{G}_{7,8,9} = -i\sigma_2 \otimes \sigma_{1,2,3} \otimes \Gamma_7 \quad (11.11)$$

and then the octonion units may be defined by:

$$\mathcal{G}_{6+n} := e_n \otimes \Gamma_7; \quad i\mathcal{G}_{11} := e_7 \otimes 1_8; \quad i\mathcal{G}_{6+n}\mathcal{G}_{11} := \hat{e}_n \otimes \Gamma_7 \quad (11.12)$$

which, with the above conventions, close the octonion algebra.

Observe that there may be other representations of the octonion units in the generators of $\mathbb{C}\ell(1, 7)$ and $\mathbb{C}\ell(1, 9)$ (see A2).

Let us now define the so-called complex octonions:

$$u_{\pm} = \frac{1}{2}(1 \pm ie_7); \quad v_{\pm}^{(n)} = \frac{1}{2}e_n(1 \pm ie_7) \quad (11.13)$$

it is known [14] that, in so far they select in octonion space the preferred direction e_7 , they define an $SU(3)$ invariant algebra (see A2), for which $v_+^{(n)}$ and $v_-^{(n)}$ transform as the (3) and $(\bar{3})$ representations of $SU(3)$ respectively, while u_+ and u_- as singlets. Furthermore $u_+, v_+^{(n)}$ and $u_-, v_-^{(n)}$ define quaternion algebras, and, following the definitions (11.4) by which $e_0 := 1_4$, they may be concisely indicated with $v_{\pm}^{(m)}$ for $m = 1, 2, 3, 0$ (since $v_{\pm}^{(0)} \equiv u_{\pm}$).

In $\mathbb{C}\ell(1, 7)$ they become:

$$u_{\pm} = \frac{1}{2}(1 \pm G_9); \quad v_{\pm}^{(n)} = \frac{1}{2}G_{4+n}(1 \pm G_9) \quad (11.14)$$

and in $\mathbb{C}\ell(1, 9)$:

$$U_{\pm} = \frac{1}{2}(1 \pm \mathcal{G}_{11}); \quad V_{\pm}^{(n)} = \frac{1}{2}\mathcal{G}_{6+n}(1 \pm \mathcal{G}_{11}), \quad (11.14')$$

again concisely expressible as $V_{\pm}^{(m)}$, with $m = 1, 2, 3, 0$. We see that they contain both the generators of the Clifford algebras and the projectors apt to insert them in the above Cartan's equations, and eq.(11.6) becomes:

$$(P_{\mu}G^{\mu} + \sum_{n=1}^3 P_{4+n}v_{\pm}^{(n)} + P_8G_8)u_{\pm}\Theta = 0, \quad (11.6')$$

which for $u_+\Theta = N$, doublet of Dirac spinors (remember that $\Gamma_a = \Gamma_a^{(0)}$) identifies with eq.(6.5) presenting the $SU(2)$ isospin internal symmetry where clearly only the quaternion subalgebra e_n is acting. Eq.(11.10) becomes:

$$(P_a\mathcal{G}^a + \sum_{n=1}^3 P_{6+n}V_{\pm}^{(n)} + P_{10}\mathcal{G}_{10})v_{\pm}\Phi = 0. \quad (11.10')$$

Identifying the reduced spinor $v_+\Phi$ with Θ_B of eq.(10.1):

$$\Theta_B = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{pmatrix} = \begin{pmatrix} B_1 \\ B_2 \end{pmatrix},$$

eq.(11.10') presents, (for $G_A = G_A^{(0)}$), an internal symmetry $SU(2)$ acting on the fermion doublets B_1 and B_2 and another $SU(2)$ is present in the terms $P_5\mathcal{G}_5 + P_6\mathcal{G}_6 + P_7\mathcal{G}_7$ acting on the fermions b_1, b_2 and b_3, b_4 (for $\Gamma_a = \Gamma_a^{(0)}$).

Here an internal symmetry $SU(3)$ may emerge, and it will be flavor, since it has isospin $SU(2)$ as a sub-symmetry in the reduced space. In fact observe that $V_{\pm}^{(n)}$ for $n = 1, 2, 3$ (saturated by P_{6+n}) transform as (3) of $SU(3)$. But it should also be possible to obtain the known Gell-Mann representation of the pseudo-octonion algebra [15] by acting with the 3 operators corresponding to $V_{\pm}^{(n)}$ on Cartan-standard spinors [1], or equivalently, on the vacuum of a Fock representation of spinor space as in ref.[4], to obtain, as minimal left ideals, 3 spinors representing quarks, as will be discussed elsewhere. In this way the first two sums in eq.(10.3) could be realized.

11.3 $SU(3)$ color

But there is another $SU(3)$ invariant algebra, contained in $\mathbb{C}\ell(1, 7)$ and $\mathbb{C}\ell(1, 9)$. In fact suppose that, as discussed in Chapter 7, a local transformation, generated by G_7G_8 , gives rise to a gauge potential then eq.(11.6) for $\mathbb{C}\ell(1, 7)$ may be written in four equivalent forms as eq.(6.15) in section 6.3:

$$\left[(P^\mu - A_{(m)}^\mu)G_\mu^{(m)} + \sum_{j=5}^8 P_j G_j^{(m)} \right] (1 \pm G_9)\Theta^{(m)} = 0, \quad m = 1, 2, 3, 0 \quad (11.6'')$$

where

$$A_\mu^{(m)} = \tilde{\Theta}^{(m)} G_\mu^{(m)} \Theta^{(m)} \quad (11.15)$$

and $G_A^{(m)}$ are the generators of $\mathbb{C}\ell(1, 7)$, with $\Theta^{(m)}$ the corresponding spinors as given in A1, eqs.(A1.10) to which the arguments of section 6.3 may be applied. Take now $m = 2$:

$$G_\mu^{(2)} = \sigma_2 \otimes \Gamma_\mu \quad (11.16)$$

and substitute Γ_μ with the 3 generators given in (A1.7): $\Gamma_\mu^{(n)} = \sigma_n \otimes \gamma_\mu$, where $n = 1, 2, 3$, then define:

$$\begin{aligned} -iG_\mu^{(2,n)} &= \begin{pmatrix} 0 & -\sigma_n \\ \sigma_n & 0 \end{pmatrix} \otimes \gamma_\mu := e_n \otimes \gamma_\mu \\ iG_9^{(2)} &= i\sigma_3 \otimes 1_8 = e_7 \otimes 1_4 \\ G_\mu^{(2,n)} G_9^{(2)} &:= e_n e_7 \otimes \gamma_\mu = \hat{e}_n \otimes \gamma_\mu \end{aligned} \quad (11.17)$$

and with them define the $SU(3)$ invariant complex octonion algebra:

$$u_{\pm} = \frac{1}{2}(1 \pm G_9); \quad v_{\mu\pm}^{(m)} = \frac{1}{2} e_m \otimes \gamma_\mu (1 \pm G_9). \quad (11.18)$$

Observe that for $m = 0, e_0 = 1_4$, and then we may interpret it as representing $G_\mu^{(2,0)}$ by which the spinor $u_+\Theta$ is a doublet of Dirac spinors and $v_{\mu\pm}^{(0)}$ as well as u_{\pm} transform as a singlet for $SU(3)$.

We may now introduce these in eq.(11.6'') which, taking into account of (11.6'), will become:

$$\left[(P^\mu + iA_{(2,m)}^\mu)v_{\mu\pm}^{(m)} + \sum_{n=1}^3 P_{4+n}v_{\pm}^{(n)} + P_8G_8 \right] u_{\pm}\Theta^{(2,m)} = 0. \quad (11.6''')$$

These equations identify with those discussed in Chapter 6 and therefore contain both the $SU(2)$ isospin internal symmetry and the $SU(2)_L$ one of the electroweak model (in the dynamical terms).

In a similar way from eq.(11.10) for $\mathcal{C}\ell(1,9)$ we obtain:

$$\left[(P^\mu - A_{(m)}^\mu)\mathcal{G}_\mu^{(m)} + \sum_{j=5}^{10} P_j\mathcal{G}_j^{(m)} \right] (1 \pm \mathcal{G}_{11})\Phi^{(m)} = 0 \quad m = 1, 2, 3, 0 \quad (11.10'')$$

and we will take:

$$\begin{aligned} -i\mathcal{G}_\mu^{(2,n)} &= \begin{pmatrix} 0 & -G_\mu^{(n)} \\ G_\mu^{(n)} & 0 \end{pmatrix} := e_n \otimes \Gamma_\mu \\ i\mathcal{G}_{11}^{(2)} &= e_7 \otimes 1 \\ \mathcal{G}_\mu^{(2,n)}\mathcal{G}_{11}^{(2)} &= \hat{e}_n \otimes \Gamma_\mu \end{aligned} \quad (11.19)$$

with the $SU(3)$ invariant algebra

$$U_{\pm} = \frac{1}{2}(1 \pm \mathcal{G}_{11}); \quad V_{\mu\pm}^{(m)} = \frac{-i}{2}\mathcal{G}_{\mu\pm}^{(2,m)}(1 \pm \mathcal{G}_{11}) = \frac{1}{2}e_m \otimes \Gamma_\mu(1 \pm \mathcal{G}_{11}) \quad (11.20)$$

We may then introduce in Cartan's equation (11.10'') the $SU(3)$ invariant algebras and it becomes:

$$\begin{aligned} &\left[(P^\mu + iA_{(2,m)}^\mu)V_{\mu\pm}^{(m)} + P_5\mathcal{G}_5^{(2,m)} + P_6\mathcal{G}_6^{(2,m)} \right. \\ &\quad \left. + \sum_{n=1}^3 P_{6+n}V_{\pm}^{(n)} + P_{10}\mathcal{G}_{10}^{(2,m)} \right] U_{\pm}\Phi^{(2,m)} = 0. \end{aligned} \quad (11.10''')$$

Observe that eq.(11.10'''), after a dimensional reduction reduces to eq.(11.6''') above containing $SU(2)$ isospin and $SU(2)_L$ electroweak, therefore if, as discussed above (for eq.(11.10')) the $SU(3)$ acting on $U_+ = V_+^{(0)}$ and $V_+^{(n)}$ represents flavour, the one acting on $V_{\mu+}^{(0)}$ and $V_{\mu+}^{(n)}$ could represent color, for which $A_\mu^{(2,n)}$ with $n = 1, 2, 3$ could represent the colored gluons. Observe that, in this interpretation the 3 colors are characterized by e_n ; that is by the quaternion subalgebra of octonions, that is by their 3 imaginary units (Pauli matrices).

In the dimensional reduced (11.6''') $V_{\mu+}^\mu$ identifies with v_+^μ giving rise to the electroweak $SU(2)_L$ which would then appear as correlated with $SU(3)$ -color in a parallel way as $SU(2)$ isospin may be considered as a subgroup of $SU(3)$ -flavour. However, $SU(2)_L$ is not a subgroup since it does not refer to the same interactions (strong) of $SU(3)$ -color.

Also for $SU(3)$ -color the arguments on the possibility of deriving a 3×3 representations of the corresponding pseudo-octonion algebras may be adopted.

Observe that the proposed geometrical interpretation of color could give a geometrical explanation on why only uncolored fermions may appear as free particles; that is, on why only $SU(3)$ -color singlets, are observable. In fact $V_{\mu+}^{(0)} = 1 \otimes \Gamma_{\mu}(1 + \mathcal{G}_{11})$ are $SU(3)$ singlets and correspond to observable multiplets of Dirac spinors (possibly trilinear in quarks); while the colored triplets $V_{\mu+}^{(n)}$ correspond also to Weyl multiplets which are unobservable since they obey coupled Dirac equations and transform in each other for space-time reflections. This is also evident for the $SU(2)_L$ subgroups of the electroweak model where the chirally projected electron: $e_L = \frac{1}{2}(1 + \gamma_5)e$ (or proton or neutron) cannot be observed as a free fermion. There is also a correlation of the $A_{(2,j)}^{\mu} - SU(3)$ (with $j = 1, 2, 3$) gluons with the W_{μ}^{\pm} and $Z_{\mu}^0 - SU(2)_L$ -mesons to be further analysed, the former bearing the strong charge while the latter (bilinear in twistors [8]) the electroweak one.

In all this the quaternion subalgebra $1, e_1, e_2, e_3$, of the octonion algebra plays a basic role, as it played in the determination of families in Chapter 10. Therefore quaternions seem to be the unique origin of isospin, electroweak, 3 families, and 3 colors.

As mentioned in the Introduction our spinor approach was at first proposed as an alternative to the traditional one in space-time, which is also postulated to be ten dimensional and lorentzian, where however the concept of geometrical point-event is substituted by that of strings or superstrings, which also allowed the extension of the theory to the possibly quantizable gravitational field. The two approaches might not however be as alternative as they might appear at first sight, and not only because we arrived, through spinor-geometrical arguments, at the same dimension and signature, even if in momentum space, but specially because also strings and perhaps even superstrings might naturally originate (bilinearly) from simple or pure spinors as we will briefly show.

12 STRINGS FROM SPINORS

The central role which null-vectors and null-lines (lines with null tangent) played in the last two centuries in the development of geometry, in the frame of complex analysis, emerged from the well-known Enneper-Weierstrass parametrization of minimal surfaces in \mathbb{R}^3 , in the form:

$$X_j(u, v) = X_j(0, 0) + \operatorname{Re} \int_c^{u+iv} Z_j(\alpha) d\alpha; \quad j = 1, 2, 3 \quad (12.1)$$

where $X_j(u, v)$ are the orthonormal coordinates of the points of a surface, which is minimal provided $Z_j(\alpha)$ are the holomorphic coordinates of a null \mathbb{C}^3 -vector, and c is any path in the complex plane (u, v) starting from the origin. The correlation with spinors associated with null \mathbb{C}^3 -vectors, are given by eqs.(3.1'), where $Z_j = \frac{1}{2} \langle \varphi^t \epsilon \sigma_j \varphi \rangle$ satisfy identically eq.(3.2). It was shown in ref.[16] that, by considering φ as a Weyl spinor associated with $\mathbb{C}\ell(3, 1)$ eq.(12.1) may easily be extended to

$\mathbb{R}^{3,1}$. For a Majorana spinor associated with $\mathcal{Cl}(3, 1) = R(4)$ the corresponding equation gives the representation of a string in the form:

$$X_\mu(\sigma, \tau) = X_\mu(0, 0) + \int_0^{\sigma+\tau} t_\mu^+(\alpha) d\alpha \int_0^{\sigma-\tau} t_\mu^-(\beta) d\beta; \quad \mu = 0, 1, 2, 3 \quad (12.2)$$

where t_μ^\pm are real, null vectors bilinearly constructed in terms of Majorana spinors. It was further shown [17] that the above formalism may be extended to higher dimensional Clifford algebras and corresponding spinor spaces and that, whenever real Majorana spinors are admitted, strings will be naturally obtained as integrals of bilinear null vectors in terms of them, which applies in particular to the case of $\mathcal{Cl}(9, 1)$, in which frame the string approach to the gravitational field is notoriously contained. Propositions 1 and 2 of the present paper may be adopted also in the case of strings such that imbedding simple spinor spaces in higher dimensional ones implies the corresponding imbedding of strings.

In this way the approach presented in this paper could not only be compatible with, but even be at the origin of string theory, in accordance with the Cartan's conception of simple spinor geometry on which this paper is based. In fact Cartan conjectured that the fundamental geometry of nature be that of simple spinors, out of which euclidean geometry derives, in so far simple spinors bilinearly generate null vectors and sums of null vectors may give the ordinary non null vectors of euclidean geometry [1]. Now the integrals above, defining strings, may be conceived as continuous sums of null vectors and therefore, in the spirit of Cartan, they could well be thought as the intermediate step between simple spinor- and euclidean-geometry where then they could well have to substitute the purely euclidean concept of point-event.

A bilinear parametrization of covariant strings and superstrings theories in terms of Majorana-Weyl spinors associated with $\mathcal{Cl}(1, 9)$ was proposed in refs.[18], [19]. The general solutions of the equations of motion are obtained through the use of octonions, which might render transparent, through the triality automorphism, also the geometrical origin of supersymmetry. Recently, pure spinors have been successfully adopted in superstring theory [20].

13 FURTHER GEOMETRICAL ASPECTS AND CONCLUDING REMARKS

We attempted to show how the elegant geometry of simple or pure spinors could be helpful for throwing some light on several aspects of the tantalizing world of the elementary constituents of matter and, if correlated with the division algebras, naturally emerging from that geometry, could plainly explain the origin of charges, internal symmetry groups and families. The strict correlation of division algebras with elementary particle physics has been also shown by G.M. Dixon [21].

In this preliminary approach we concentrated our attention on the main features of Cartan's conception of simple spinors as isomorphic (up to a sign) to maximal totally null planes defining projective quadrics and corresponding compact manifolds (spheres) all imbedded one in the other up to that in a ten dimensional lorentzian momentum space. The basic role of the geometry of simple or pure

spinors is manifested not only by the fact that it allows the simultaneous embeddings of spinor spaces and of the corresponding null vector spaces, but also by the fact that the equations of motions, originally in momentum spaces, may be written solely in spinor spaces, and the necessary condition for the existence of solutions ($p_a p^a = 0$) become identities, if, as suggested by Cartan, the components of momenta are expressed bilinearly in terms of spinors, if these are simple or pure; and also the equations of motion may become identities in spinor spaces. That is, they are true in the whole simple or pure spinor space independently from momentum space or space-time. This could then assign a purely geometrical, that is spinorial, origin to the equations of motion; and then to the laws they represent, which are then to be conceived, so to say, outside space-time: in the realm of simple or pure spinor-geometry. Obviously if a p_μ -dependent (or x_μ -dependent) spinor-solution is inserted in the equations of motion they also become identities, however in the whole momentum space; meaning the validity for any p or, equivalently, the validity of the evolution of the phenomena contemplated by the laws, for the whole space-time. Maxwell's equations are an example: the electromagnetic tensor $F_{\mu\nu}^\pm$ derives, bilinearly, from the Weyl's spinors $\varphi_+ = \frac{1}{2}(1 + \gamma_5)\psi$ and $\varphi_- = \frac{1}{2}(1 - \gamma_5)\psi$ (see eq.(3.13)), which determine then its properties (as noted already by Cartan [1]), while their solutions: $F_{\mu\nu}^\pm(x, t)$ well represent propagation of light both here, now, and on the distant galaxies as they were billions of years ago. The same is true for the Cartan's equations⁴ in this paper where momenta p_μ (or space-time coordinates x_μ , that is t_μ^\pm in eq.(12.2)) are bilinear functions of spinors and as such make identities both the necessary conditions for the existence of solutions and the equations of motion, while their solutions are spinors, functions of x_μ , making them identities in space-time. Then simple spinors determine bilinearly p_μ (or x_μ) while Cartan's equations represent the physical laws, and their solutions, the universal evolution of the phenomena. In this way the origin of the physical laws ruling the behaviour of the elementary constituents of matter: the fermions⁵, might naturally lay in simple spinor geometry, which, in turn, following Cartan, may be conceived as the elementary constituent of euclidean geometry.

The geometry of spinors we dealt with is however very rich and presents further aspects like the constraint equations for simple spinors and the triality symmetry for eight dimensional spaces, which, while geometrically fundamental, could also be of interest for comparing the mathematical results with the physical world. Specially, the constraint-equations, correlated with simplicity, and then allowing the

⁴Only Cartan's equations for simple or pure spinors appear to present this double possibility of becoming identities: both in spinor space (if momenta are bilinear functions of spinors) and in momentum space or space-time (if spinors are functions of momenta or space-time; that is solutions). This double possibility allows an epistemological correlation between the laws of physics (the equations) and the consequent phenomena (their solutions): the former to be conceived outside space-time "are" in Parmenides conception of "to be" which "neither was nor will be but always is". The phenomena instead universally evolve in space-time, as foreseen by the solutions of those equations or laws, and then "become" well representing Eraclito's "panta rei". Cartan's simple spinor geometry could then well represent an example of synthesis between these two apparently antithetic philosophies.

⁵Bosons, in this frame, result bi- or multi-linear in spinors and represent either composite states, with inner structure, as notoriously happens in nuclear physics (helium) or just new euclidean scalar, vector or tensor fields, without inner structure, as in the case of the electromagnetic gauge potential and, presumably, of the colored gauge gluons.

whole construction, through Propositions 1 and 2, might play important roles (as recent results seem to indicate [20]) and could also furnish interesting indications on the stability of the proton [8] and on the possibility of theoretical predictions, which were not dealt with in the present preliminary paper, to be completed with a study of the physical meaning of the steadily increasing values of the invariant masses characterizing the spheres imbedded in each other, presumably accompanied by the increasing values of the corresponding charges, which also should be geometrically determined as will be discussed elsewhere.

Should this approach have further confirmations in the physical world, then the old Cartan's conjecture on the fundamental role of simple spinor geometry, at the origin of euclidean geometry, would not only obtain a strong support, but it could also be extended to the origin of the geometry of quantum mechanics, in so far, the Cartan's equations defining simple spinors, may be interpreted as equations of motion for fermions, the most elementary constituents of matter, and they are quantum equations (in first quantization, up to the definition of Planck's constant). A remarkable aspect of these equations is that, at first, they naturally appear in momentum space, sometimes called the space of velocities; after all a natural space for the description of elementary motions. If furthermore, as it would here appear, in momentum space several of the mysterious aspects of the elementary constituents of matter, might be easily explained and well understood, in purely geometrical (spinorial) form, one could also hope that in this space (of velocities or rather of elementary motions) several of the difficulties, which notoriously hinder the understanding of quantum mechanics, when described in space-time, could be substantially milder if not disappear.

One could then be induced to conjecture that while space-time remains the ideal arena for the euclidean description of the classical form of mechanics (of astronomy), it is momentum space the ideal one for the description, by means of the elementary form of geometry: the Cartan's one of simple spinors, of the elementary form of mechanics: quantum mechanics (of fermions); the two being possibly correlated by conformal inversions or reflections [22].

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APPENDICES

A1 CLIFFORD ALGEBRAS, SPINORS, OPTICAL VECTORS, CARTAN'S EQUATIONS: FROM $\mathbb{C}\ell(1, 3)$ TO $\mathbb{C}\ell(1, 9)$

Notations

Indices:

$$n = 1, 2, 3; \quad \mu = 1, 2, 3, 0; \quad a = \mu, 5, 6; \quad A = a, 7, 8; \quad \alpha = A, 9, 10. \quad (\text{A1.1})$$

Spinors:

$$\begin{aligned} \psi \in S_4 : \mathbb{C}\ell(1, 3) &= \text{End } S_4; & \tilde{\psi} &:= \psi^\dagger \gamma_0, \\ \Psi \in S_8 : \mathbb{C}\ell(1, 5) &= \text{End } S_8; & \tilde{\Psi} &:= \Psi^\dagger \Gamma_0, \\ \Theta \in S_{16} : \mathbb{C}\ell(1, 7) &= \text{End } S_{16}; & \tilde{\Theta} &:= \Theta^\dagger G_0, \\ \Phi \in S_{32} : \mathbb{C}\ell(1, 9) &= \text{End } S_{32}; & \tilde{\Phi} &:= \Phi^\dagger \mathcal{G}_0. \end{aligned} \quad (\text{A1.2})$$

$\mathbb{C}\ell(1, 3)$; generators γ_μ :

$$\gamma_\mu : \quad \gamma_n = \begin{pmatrix} 0 & -\sigma_n \\ \sigma_n & 0 \end{pmatrix} = -i\sigma_2 \otimes \sigma_n; \quad \gamma_0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \sigma_1 \otimes 1_2; \quad (\text{A1.3})$$

volume element

$$\gamma_5 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \sigma_3 \otimes 1_2. \quad (\text{A1.4})$$

Optical vectors:

$$p_\mu^\pm = \tilde{\psi}(1 \pm \gamma_5)\gamma_\mu\psi, \quad (\text{A1.5})$$

(the factor $\frac{1}{4}$ was eliminated for brevity: here and in the following we are dealing with null vectors which may anyhow be arbitrarily scaled.)

Cartan's equation:

$$p_\mu^\pm \gamma^\mu (1 \pm \gamma_5)\psi = 0 \quad (\text{A1.6})$$

$\mathbb{C}\ell(1, 5)$; generators Γ_a :

$$\begin{array}{l}
\text{Weyl:} \\
\text{Pauli:} \\
\text{Dirac:}
\end{array}
\left\{ \begin{array}{l}
\Gamma_\mu^{(1)} = \sigma_1 \otimes \gamma_\mu; \quad \Gamma_5^{(1)} = -i\sigma_1 \otimes \gamma_5, \Gamma_6^{(1)} = -i\sigma_2 \otimes 1_4, \Gamma_7^{(1)} = \sigma_3 \otimes 1_4 \\
\Gamma_\mu^{(2)} = \sigma_2 \otimes \gamma_\mu; \quad \Gamma_5^{(2)} = -i\sigma_2 \otimes \gamma_5, \Gamma_6^{(2)} = -i\sigma_1 \otimes 1_4, \Gamma_7^{(2)} = \sigma_3 \otimes 1_4 \\
\Gamma_\mu^{(3)} = \sigma_3 \otimes \gamma_\mu; \quad \Gamma_5^{(3)} = -i\sigma_1 \otimes 1_4, \Gamma_6^{(3)} = -i\sigma_2 \otimes 1_4, \Gamma_7^{(3)} = \sigma_3 \otimes \gamma_5 \\
\Gamma_\mu^{(0)} = 1_2 \otimes \gamma_\mu; \quad \Gamma_5^{(0)} = -i\sigma_1 \otimes \gamma_5, \Gamma_6^{(0)} = -i\sigma_2 \otimes \gamma_5, \Gamma_7^{(0)} = \sigma_3 \otimes \gamma_5
\end{array} \right. \quad (\text{A1.7})$$

Optical vectors (Corollary 2):

$$p_a^\pm : p_\mu^\pm = \tilde{\psi} \gamma_\mu \psi; \quad p_5^\pm = -i\tilde{\psi} \gamma_5 \psi, \quad p_6^\pm = \pm \tilde{\psi} \psi \quad (\text{A1.8})$$

or:

$$p_a^\pm = \tilde{\Psi} \Gamma_a (1 \pm \Gamma_7) \Psi \quad (\text{A1.8}')$$

Cartan's equation:

$$p_a^\pm \Gamma^a (1 \pm \Gamma_7) \Psi = 0 \quad (\text{A1.9})$$

$\mathbb{C}\ell(1, 7)$; generators G_A :

$$\begin{array}{l}
\text{Weyl:} \\
\text{Pauli:} \\
\text{Dirac:}
\end{array}
\left\{ \begin{array}{l}
G_a^{(1)} = \sigma_1 \otimes \Gamma_a; \quad G_7^{(1)} = -i\sigma_1 \otimes \Gamma_7, G_8^{(1)} = -i\sigma_2 \otimes 1_8, G_9^{(1)} = \sigma_3 \otimes 1_8 \\
G_a^{(2)} = \sigma_2 \otimes \Gamma_a; \quad G_7^{(2)} = -i\sigma_2 \otimes \Gamma_7, G_8^{(2)} = -i\sigma_1 \otimes 1_8, G_9^{(2)} = \sigma_3 \otimes 1_8 \\
G_a^{(3)} = \sigma_3 \otimes \Gamma_a; \quad G_7^{(3)} = -i\sigma_1 \otimes 1_8, G_8^{(3)} = -i\sigma_2 \otimes 1_8, G_9^{(3)} = \sigma_3 \otimes \Gamma_7 \\
G_a^{(0)} = 1_2 \otimes \Gamma_a; \quad G_7^{(0)} = -i\sigma_1 \otimes \Gamma_7, G_8^{(0)} = -i\sigma_2 \otimes \Gamma_7, G_9^{(0)} = \sigma_3 \otimes \Gamma_7
\end{array} \right. \quad (\text{A1.10})$$

Optical vectors (Corollary 2):

$$P_A^\pm : P_a^\pm = \tilde{\Psi} \Gamma_a \Psi; \quad P_7^\pm = -i\tilde{\Psi} \Gamma_7 \Psi, \quad P_8^\pm = \pm \tilde{\Psi} \Psi \quad (\text{A1.11})$$

or:

$$P_A^\pm = \tilde{\Theta} G_A (1 \pm G_9) \Theta \quad (\text{A1.11}')$$

Cartan's equation:

$$P_A^\pm G^A (1 \pm G_9) \Theta = 0. \quad (\text{A1.12})$$

If

$$\Theta = \begin{pmatrix} \Psi_+ \\ \Psi_- \end{pmatrix} \quad (\text{A1.13})$$

Cartan's equation:

$$(P_a \Gamma^a - iP_7 \Gamma^7 \pm P_8) \Psi_\pm = 0. \quad (\text{A1.12}')$$

Taking for Γ_a the Dirac's rep. Γ_a^0 as in (A1.7) we obtain:

Cartan's equation:

$$(P_\mu \gamma^\mu - i\vec{\pi} \cdot \vec{\sigma} \otimes \gamma_5 \pm P_8) \Psi_\pm = 0 \quad (\text{A1.12}'')$$

where $\pi_{1,2,3}$ are pseudo scalars of $\mathbb{R}^{3,1}$ (see eq.(6.6)), and, therefore

$$\pi_{1,2,3} \otimes \gamma_5 = P_{5,6,7} \quad (\text{A1.14})$$

are scalars of $\mathbb{R}^{3,1}$; we will indicate them with $P_{5,6,7} := s_{4+n}$, with $n = 1, 2, 3$.

If we now represent with $q_n = -i\sigma_n$ the quaternion units we obtain the Cartan's equation (A1.12'') in the quaternionic form:

$$(P_\mu \gamma^\mu + s_{4+n} q^n \pm P_8) \Psi_\pm = 0 \quad (\text{A1.12''})$$

$\mathbb{C}\ell(1, 9)$; generators \mathcal{G}_α :

$$\begin{array}{l} \text{Weyl:} \\ \text{Pauli:} \\ \text{Dirac:} \end{array} \left\{ \begin{array}{l} \mathcal{G}_A^{(1)} = \sigma_1 \otimes G_A; \quad \mathcal{G}_9^{(1)} = -i\sigma_1 \otimes G_9, \mathcal{G}_{10}^{(1)} = -i\sigma_2 \otimes 1_{16}, \mathcal{G}_{11}^{(1)} = \sigma_3 \otimes 1_{16} \\ \mathcal{G}_A^{(2)} = \sigma_2 \otimes G_A; \quad \mathcal{G}_9^{(2)} = -i\sigma_2 \otimes G_9, \mathcal{G}_{10}^{(2)} = -i\sigma_1 \otimes 1_{16}, \mathcal{G}_{11}^{(2)} = \sigma_3 \otimes 1_{16} \\ \mathcal{G}_A^{(3)} = \sigma_3 \otimes G_A; \quad \mathcal{G}_9^{(3)} = -i\sigma_1 \otimes 1_{16}, \mathcal{G}_{10}^{(3)} = -i\sigma_2 \otimes 1_{16}, \mathcal{G}_{11}^{(3)} = \sigma_3 \otimes G_9 \\ \mathcal{G}_A^{(0)} = 1_2 \otimes G_A; \quad \mathcal{G}_9^{(0)} = -i\sigma_1 \otimes G_9, \mathcal{G}_{10}^{(0)} = -i\sigma_2 \otimes G_9, \mathcal{G}_{11}^{(0)} = \sigma_3 \otimes G_9 \end{array} \right. \quad (\text{A1.15})$$

Optical vectors (Corollary 2):

$$P_A^\pm : \tilde{\Theta} G_A \Theta; \quad P_9^\pm = -i\tilde{\Theta} G_9 \Theta; \quad P_{10}^\pm = \pm \tilde{\Theta} \Theta, \quad (\text{A1.16})$$

or:

$$P_\alpha^\pm = \tilde{\Phi} \mathcal{G}_\alpha (1 \pm \mathcal{G}_{11}) \Phi \quad (\text{A1.16}')$$

Cartan's equation:

$$P_\alpha \mathcal{G}^\alpha (1 \pm \mathcal{G}_{11}) \Phi = 0. \quad (\text{A1.17})$$

If

$$\Phi = \begin{pmatrix} \Theta_+ \\ \Theta_- \end{pmatrix} \quad (\text{A1.18})$$

Cartan's equation (for the representation $G_A^{(1)}$ from (A1.10)):

$$(P_A G^A - iP_9 G^9 \pm P_{10}) \Theta_\pm = 0 = \begin{pmatrix} \pm P_{10} - iP_9 & P_a \Gamma^a - iP_7 \Gamma^7 - P_8 \\ P_a \Gamma^a - iP_7 \Gamma^7 + P_8 & \pm P_{10} + iP_9 \end{pmatrix} \Theta_\pm \quad (\text{A1.18}')$$

The non diagonal terms are of the form (A1.12'). Setting them in the form (A1.12''), and taking into account of (A1.4) we have that, in (A1.18'), after setting $\Gamma_a = \Gamma_a^{(0)}$:

$$G_{5,6,7}^{(1)}(\Gamma_a^{(0)}) := G_{5,6,7}^{(1,0)} = \begin{pmatrix} 0 & -i\sigma_{1,2,3} \\ -i\sigma_{1,2,3} & 0 \end{pmatrix} \otimes \gamma_5 = -i\sigma_1 \otimes \sigma_{1,2,3} \otimes \gamma_5 \quad (\text{A1.19})$$

If we impose that $P_{5,6,7}$ are scalars of $\mathbb{R}^{3,1}$ the γ_5 factor becomes 1: see eq.(A1.14).

For the representation $G_A^{(2)}$ from (A1.10) instead, Cartan's equation becomes:

$$\begin{pmatrix} \pm P_{10} - iP_9 & -iP_a\Gamma^a - P_7\Gamma^7 + iP_8 \\ -iP_a\Gamma^a + P_7\Gamma^7 + iP_8 & \pm P_{10} + P_9 \end{pmatrix} \Theta_{\pm} = 0 \quad (\text{A1.18}'')$$

With the same procedure as before one obtains that, this time:

$$G_{5,6,7}^{(2)}(\Gamma_a^{(0)}) := G_{5,6,7}^{(2,0)} = \begin{pmatrix} 0 & -\sigma_{1,2,3} \\ \sigma_{1,2,3} & 0 \end{pmatrix} \otimes \gamma_5 = -i\sigma_2 \otimes \sigma_{1,2,3} \otimes \gamma_5 \quad (\text{A1.20})$$

Again $\gamma_5 = 1$, for $P_{5,6,7}$ scalars.

According to Corollary 2 applied to $\mathcal{Cl}(1, 9)$ we have that the vector with components:

$$\mathcal{P}_{\alpha}^{\pm} = \tilde{\Phi}\mathcal{G}_{\alpha}\Phi; \quad \mathcal{P}_{11}^{\pm} = -i\tilde{\Phi}\mathcal{G}_{11}\Phi; \quad \mathcal{P}_{12}^{\pm} = \pm\tilde{\Phi}\Phi, \quad (\text{A1.21})$$

is null in $\mathbb{R}^{1,11}$, for Φ simple Weyl of $\mathcal{Cl}_0(1, 11)$. They give rise to Cartan's equation:

$$(\mathcal{P}_{\alpha}\mathcal{G}^{\alpha} - i\mathcal{P}_{11}\mathcal{G}^{11} \pm \mathcal{P}_{12})\phi_{\pm} = 0 = \begin{pmatrix} \pm\mathcal{P}_{12} - i\mathcal{P}_{11} & \mathcal{P}_A G^A - i\mathcal{P}_9 G^9 - \mathcal{P}_{10} \\ \mathcal{P}_A G^A - i\mathcal{P}_9 G^9 + \mathcal{P}_{10} & \pm\mathcal{P}_{12} + i\mathcal{P}_{11} \end{pmatrix} \quad (\text{A1.22})$$

where the representation $\mathcal{G}_{\alpha}^{(1)}$ was taken from (A1.15). If we now insert in it $G_A^{(0)}$ from (A1.10) we obtain:

$$\mathcal{G}_{7,8,9}^{(1)}(G_A^{(0)}) := \mathcal{G}_{7,8,9}^{(1,0)} = -i\sigma_1 \otimes \sigma_{1,2,3} \otimes \Gamma_7. \quad (\text{A1.23})$$

For the choice $\mathcal{G}_{\alpha}^{(2)}$ instead we would have obtained:

$$\mathcal{G}_{7,8,9}^{(2)}(G_A^{(0)}) := \mathcal{G}_{7,8,9}^{(2,0)} = -i\sigma_2 \otimes \sigma_{1,2,3} \otimes \Gamma_7. \quad (\text{A1.24})$$

A2 MATRIX REPRESENTATIONS OF OCTONIONS

Notations

Octonions:

$$\begin{aligned} \mathbf{o} &= e_0 p_8 + \sum_{j=1}^7 p_j e_j \\ \bar{\mathbf{o}} &= e_0 p_8 - \sum_{j=1}^7 p_j e_j \\ \mathbf{o}\bar{\mathbf{o}} &= \bar{\mathbf{o}}\mathbf{o} = \sum_{j=1}^8 p_j^2 \end{aligned} \quad (\text{A2.1})$$

where $p_j \in \mathbb{R}$, e_0 is the identity and e_1, e_2, \dots, e_7 are the octonion imaginary anti-commuting units for which we adopt the notation:

$$\begin{aligned} e_n &= \{e_1, e_2, e_3\}; \quad e_{n+3} = \{e_4, e_5, e_6\} := \hat{e}_n = \{\hat{e}_1, \hat{e}_2, \hat{e}_3\}; \\ e_{n+3} &= \hat{e}_n := e_n e_7 = -e_7 e_n. \end{aligned} \quad (\text{A2.2})$$

The octonion algebra 0:

$$\begin{aligned} e_\ell e_m &= -\delta_{\ell m} e_0 + \epsilon_{\ell mn} e_n \\ e_\ell \hat{e}_m &= -\delta_{\ell m} e_7 - \epsilon_{\ell mn} \hat{e}_n \\ \hat{e}_\ell \hat{e}_m &= -\delta_{\ell m} e_0 - \epsilon_{\ell mn} e_n \end{aligned} \quad (\text{A2.3})$$

where $\ell, m, n = 1, 2, 3$.

The quaternion subalgebra Q is:

$$Q = \{e_0, e_1, e_2, e_3\} \quad (\text{A2.4})$$

Define

$$\hat{Q} := Qe_7 = \{e_7, \hat{e}_1, \hat{e}_2, \hat{e}_3\} \quad (\text{A2.5})$$

The non associative octonion algebra 0 may be graded as follows:

$$Q = Q \oplus \hat{Q} \quad (\text{A2.6})$$

Matrix representations

Anti De Sitter Clifford algebra $\mathbb{C}\ell(2, 3)$ with generators

$$\gamma_n = \begin{pmatrix} 0 & -\sigma_n \\ \sigma_n & 0 \end{pmatrix}; \quad \gamma_0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \gamma_5 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad (\text{A2.7})$$

Define:

$$e_0 := \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}; \quad e_n := \gamma_n; \quad e_7 := i\gamma_5 = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \quad (\text{A2.8})$$

$$e_n e_7 := e_{n+3} := \hat{e}_n := \begin{pmatrix} 0 & i\sigma_n \\ i\sigma_n & 0 \end{pmatrix} \quad (\text{A2.9})$$

they close the octonion algebra either by modifying the multiplication rules of matrices [13] or by defining the product \odot as follows:

$$e_j \odot e_k := \delta_{jk} e_j e_k + (1 - \delta_{jk}) i \gamma_5 \gamma_0 e_j \gamma_0 e_k \gamma_0 \quad (\text{A2.10})$$

(on the r.h.s. products are ordinary matrix multiplications) for $j, k = 1, 2, \dots, 6$. While for e_7 the multiplication is defined by (A2.9) and obviously by $\hat{e}_n e_7 = -e_n$.

The products \odot and (A2.9) close the octonion algebra of the imaginary units e_0, e_1, \dots, e_7 .

Another matrix representation of the octonion units may be obtained by defining:

$$e_0 := \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}; \quad e_n := \begin{pmatrix} 0 & -i\sigma_n \\ -i\sigma_n & 0 \end{pmatrix}; \quad e_7 := \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}; \quad (\text{A2.8}')$$

$$e_n e_7 := e_{n+3} := \hat{e}_n = \begin{pmatrix} 0 & -\sigma_n \\ \sigma_n & 0 \end{pmatrix} \quad (\text{A2.9}')$$

after which the \odot product becomes

$$e_j \odot e_n = \delta_{jk} e_j e_k + (1 - \delta_{jk}) \gamma_0 e_j \gamma_0 e_k \gamma_0 \quad (\text{A2.10}')$$

For $j, k = 1, 2, \dots, 6$, where $\gamma_0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. The rest as before.

The associator between octonions is defined as usual with respect to the \odot products; in particular the one between the imaginary units e_j is defined by

$$(e_i, e_j, e_k) := (e_i \odot e_j) \odot e_k - e_i \odot (e_j \odot e_k) \quad (\text{A2.11})$$

and one easily finds the usual seven non zero ones.

Complex octonions

are defined by:

$$u_{\pm} = \frac{1}{2}(1 \mp ie_7) = \frac{1}{2}(1 \pm \gamma_5) : v_{\pm}^{(n)} = \frac{1}{2}(e_n \mp i\hat{e}_n) = e_n \frac{1}{2}(1 \pm \gamma_5) \quad (\text{A2.12})$$

where the equalities with the chiral projectors $\frac{1}{2}(1 \pm \gamma_5)$ are valid for both the proposed matrix representation of the units e_j .

The complex octonions satisfy to the following multiplication rules:

$$\begin{aligned} u_+^2 &= u_+; & u_+ u_- &= 0; & v_+^{(n)} u_+ &= u_- v_+^{(n)} \\ u_+ v_+^{(n)} &= v_+^{(n)} u_- = 0; & v_+^{(\ell)} v_-^{(m)} &= -\delta_{\ell m} u_0; & v_+^{(\ell)} v_+^{(m)} &= \epsilon_{\ell mn} v_-^{(n)}. \end{aligned} \quad (\text{A2.13})$$

and the ones obtained substituting $u_+, v_+^{(n)}$ with $u_-, v_-^{(n)}$ and viceversa. The corresponding algebra is $SU(3)$ invariant if $(v_+^{(1)} v_+^{(2)} v_+^{(3)})$ and $(v_-^{(1)} v_-^{(2)} v_-^{(3)})$ transform like the (3) and $\bar{(3)}$ representations of $SU(3)$ and u_+, u_- like its singlet representation. This $SU(3)$ is the subgroup of the G_2 group of automorphism of the octonion algebra since e_7 , that is $i\gamma_5$ in our case, is chosen as a preferred direction.

Octonions in $\mathbb{C}\ell(1, 7) - \mathbb{C}\ell(1, 9)$ and in the corresponding Cartan's equations

a) In interaction terms

$\mathbb{C}\ell(1, 7)$. Consider now eq.(A1.20) for the generators $G_5^{(2,0)}, G_6^{(2,0)}, G_7^{(2,0)}$ and the definition (A2.8) of e_n we may write it now in the form

$$G_{4+n} = e_n \otimes \gamma_5 \quad (\text{A2.14})$$

Now G_5, G_6, G_7 are the generators which, in eqs.(A1.12'') and (A1.12'''), give origin to the $SU(2)$ symmetry of isospin. They may be then, together with the unit e_0 ,

be identified with the quaternionic subalgebra Q of eq.(A2.4) of octonions. In the frame of $\mathbb{C}\ell(1, 7)$ we may also identify the generators of $\hat{Q} = Qe_7$.

In fact since $G_9^{(2)} = \sigma_3 \otimes 1_8$ (see (A1.10)) we define

$$iG_9 = e_7 \otimes 1_4$$

and

$$(A2.15)$$

$$iG_{4+n}G_9 = e_n e_7 \otimes \gamma_5 = \hat{e}_n \otimes \gamma_5.$$

Again the γ_5 factor equals one if $P_{5,6,7}$ are scalars of $\mathbb{R}^{3,1}$. These seven elements of $\mathbb{C}\ell(1, 7)$ will then close the octonion algebra if we define it with the product \odot , defined in (A2.10). If we start from the definition (A1.19) of $G_{5,6,7}$ and adopt definitions (A2.8') for the octonion units e_j , we obtain again (A2.15) which closes the octonion algebra however with (A2.10').

Let us now define the complex octonions for $\mathbb{C}\ell(1, 7)$. They are:

$$u_{\pm} = \frac{1}{2}(1 \pm G_9), \quad v_{\pm}^{(n)} = \frac{1}{2} G_{4+n}(1 \pm G_9). \quad (A2.16)$$

They all contain the projectors $\frac{1}{2}(1 \pm G_9)$ which operate the dimensional reduction; and their algebra is associative, therefore we may operate with them, normally in spinor space.

$\mathbb{C}\ell(1, 9)$. The procedure may be repeated, starting from eqs.(A1.25) and (A1.26) and we obtain:

$$\begin{aligned} G_{6+n} &= e_n \otimes \Gamma_7, \quad i\mathcal{G}_{11} = e_7 \otimes 1_8, \\ i\mathcal{G}_{6+n}\mathcal{G}_{11} &= e_n e_7 \otimes \Gamma_7 = \hat{e}_n \otimes \Gamma_7, \end{aligned} \quad (A2.17)$$

while the complex octonions are

$$u_{\pm} = \frac{1}{2}(1 \pm \mathcal{G}_{11}), \quad v_{\pm}^{(n)} = \frac{1}{2} \mathcal{G}_{6+n}(1 \pm \mathcal{G}_{11}) \quad (A2.18)$$

b) In dynamical terms

We have seen in section 10 that the possible families of baryon- and lepton-pairs which may be obtained through dimensional reduction of Θ_B and $\Theta_{\mathcal{L}}$ to fermion doublets are characterized by the representations of Γ_{μ} . Precisely

$$\Gamma_{\mu}^{(1,2)} = \sigma_{1,2} \otimes \gamma_{\mu}; \quad \Gamma_{\mu}^{(3)} = \sigma_3 \otimes \gamma_{\mu}; \quad \Gamma_{\mu}^{(0)} = 1_2 \otimes \gamma_{\mu}, \quad (A2.19)$$

characterize Weyl, Pauli, Dirac doublets respectively where γ_{μ} were generators of $\mathbb{C}\ell(3, 1)$. With this convention we may then adopt the first four generators of $G_{\mu}^{(2)}$ of $\mathbb{C}\ell(1, 7)$, for the first 3 Γ_{μ} in (A2.19), to be:

$$-G_{\mu}^{(2,n)} = -i\sigma_2 \otimes \Gamma_{\mu}^{(n)} = \begin{pmatrix} 0 & -\sigma_n \\ \sigma_n & 0 \end{pmatrix} \otimes \gamma_{\mu} := e_n \otimes \gamma_{\mu} \quad (A2.20)$$

where we have adopted the representation (A2.7) of e_n . Defining now

$$iG_9^{(2)} := i\sigma_3 \otimes 1_8 = e_7 \otimes 1_4$$

(A2.20')

and

$$G_\mu^{(2,n)} G_9^{(2,n)} := \hat{e}_n \otimes \gamma_\mu$$

we obtain in $\mathbb{C}\ell(1, 7)$ another octonion algebra. The complex octonions will be

$$u_\pm = \frac{1}{2}(1 \pm G_9); \quad v_\pm^{(n)} = G_\mu^{(2,n)}(1 \pm G_9) \quad (\text{A2.21})$$

The same algebras may be also easily found in $\mathbb{C}\ell(1, 9)$ with the substitution of γ_μ with Γ_μ and of G_9 with \mathcal{G}_{11} .

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